Question

Derive the Cauchy-Riemann equations as necessary conditions for the function

$$f(z) = u(x, y) + iv(x, y), \qquad (z = x + iy)$$

to be differentiable as a function of a complex variable z. State sufficient conditions involving the Cauchy-Riemann equations for f to be differentiable. Hence find the points where the function

$$f(z) = \begin{cases} \dfrac{xy^4 - ix^4y}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is differentiable.
At the points at which f is differentiable calculate the derivative $f'(z)$.

Answer

If f is differentiable at $z = x + iy$,

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{u(x + h, y) + iv(x + h, y) - u(x, y) - iv(x, y)}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

Also

$$\begin{aligned} f'(z) &= \lim_{k \to 0} \frac{u(x, y + k) + iv(x, y + k) - u(x, y) - iv(x, y)}{ik} \\ &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \end{aligned}$$

so $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

Sufficient conditions for differentiability at z are that the Cauchy-Riemann equations should be satisfied at z, and that the partial derivatives should exist in a neighbourhood of (x, y) and be continuous at (x, y).

Now $u = \dfrac{xy^4}{x^2+y^2} \qquad v = \dfrac{-x^4y}{x^2+y^2}$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{(x^2+y^2)y^4 - xy^4(2x)}{(x^2+y^2)^2} = \frac{y^6 - x^2y^4}{(x^2+y^2)^2} \\
\frac{\partial v}{\partial y} &= \frac{(x^2+y^2)(-x^4) + x^4y(2y)}{(x^2+y^2)^2} = \frac{y^2x^4 - x^6}{(x^2+y^2)^2} \\
\frac{\partial u}{\partial y} &= \frac{(x^2+y^2)4xy^3 - xy^4(2y)}{(x^2+y^2)^2} = \frac{4x^3y^3 + 2xy^5}{(x^2+y^2)^2} \\
-\frac{\partial v}{\partial x} &= -\frac{(x^2+y^2)(-4x^3y) + x^4y(2x)}{(x^2+y^2)^2} = \frac{4x^3y^3 + 2x^5y}{(x^2+y^2)^2}
\end{aligned}$$

For the Cauchy-Riemann equations to be satisfied we require

$$\begin{aligned}
y^6 - x^2y^4 &= y^2x^4 - x^6 \qquad (1) \\
4x^3y^3 + 2xy^5 &= 4x^3y^3 + 2x^5y \qquad (2)
\end{aligned}$$

From (2) $\quad 2xy^5 = 2x^5y$
so for $xy \neq 0 \quad y^4 = x^4 \quad$ i.e. $y = \pm x$
this also satisfies (1).
Now from (1) $\quad x = 0 \Rightarrow y = 0$ and $y = 0 \Rightarrow x = 0$
So if $y = \pm x \neq 0$ the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous if $(x, y) \neq (0, 0)$, so f is differentiable at $x(1 \pm i)$ for $x \neq 0$.
Now consider $z = 0$. Let $z = re^{i\theta}$
For $r \neq 0 \quad f(re^{i\theta}) = \dfrac{(r^5\cos\theta\sin^4\theta - ir^5\cos^4\theta\sin\theta)}{r^2}$

so $\left|\dfrac{f(re^{i\theta}) - f(0)}{re^{i\theta} - 0}\right| = r^2|\cos\theta\sin^4\theta - i\cos^4\theta\sin\theta|$

$\leq 2r^2 \quad \to 0$ as $r \to 0$

So $f'(0) = 0$

$f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{y^2x^4 - x^6}{(x^2+y^2)^2} - i\dfrac{4x^3y^3 + 2x^5y}{(x^2+y^2)^2}$

so when $x = y \quad f'(z) = -i\dfrac{6x^6}{4x^4} = -i\dfrac{3}{2}x^2$

when $x = -y \quad f'(z) = i\dfrac{6x^6}{4x^4} = i\dfrac{3}{2}x^2$