Vector Algebra and Geometry

Differentiation of Vectors

Vector - valued functions of a real variable

We have met the equation of a straight line in the form

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

$\mathbf{r}$ therefore varies with the real variables t; corresponding to each value of t we have a different vector $\mathbf{r}$ is a function of t.
$\mathbf{r}$ need not be a position vector. For example if we have a curve $mR^2x = x(t)$, $y = y(t)$, $\mathbf{r}(t)$ could denote the unit vector in the direction of the tangent to the curve at the point t. $\mathbf{r}$ would again vary as t varies. Vector field exs. magnetic field velocity field etc. In three dimensions if we have a vector function $\mathbf{a}(t)$ then we can express it in terms of its components, $\mathbf{a}(t) = a_1(t)\mathbf{i} + a_2(t) + a_3(t)\mathbf{k}$ in cartesian. $= a_1(t)$, $a_2(t)$, $a_3(t)$ are now function: $\mathbf{R} \to \mathbf{R}$
By analogy with functions $\mathbf{R} \to \mathbf{R}$ we investigate differentiability of a vector function$\mathbf{a}(t)$ by investigating the quotient

$$\frac{\mathbf{a}(t+h) - \mathbf{a}(t)}{h}$$

If this vector has a limit as $h \to 0$we say that $\mathbf{a}$ is differentiable at t

$$\frac{\mathbf{a}(t+h) - \mathbf{a}(t)}{h} = \frac{a_1(t+h) - a_1(t)}{h}\mathbf{i} + \frac{a_2(t+h) - a_2(t)}{h}\mathbf{j} + \frac{a_3(t+h) - a_3(t)}{h}\mathbf{k}$$

If each of the components has a limit as does LHS conversely if LHS has a limit, then taking the scalar product in turn with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ proves that each components has a limit.
Thus $\mathbf{a}(t)$ is differentiable if and only if its components are differentiable, and

$$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\mathbf{i} + \frac{da_2}{dt}\mathbf{j} + \frac{da_3}{dt}\mathbf{k}$$

Rules for differentiation

1. If λ, μ are constants then

$$\frac{d}{dt}(\lambda\mathbf{a}(t) + \mu\mathbf{b}(t)) = \lambda\frac{d\mathbf{a}}{dt} + \mu\frac{d\mathbf{b}}{dt}$$

2. If $\phi(t)$ is a function $\mathbf{R} \to \mathbf{R}$ then

$$\frac{d}{dt}(\phi(t)\mathbf{a}(t)) = \phi\frac{d\mathbf{a}}{dt} + \frac{d\phi}{dt}\mathbf{a}$$

3.
$$\frac{d}{dt}(\mathbf{a}(t) \cdot \mathbf{b}t) = a \cdot \frac{dbfb}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{b}$$

4.
$$\frac{d}{dt}(\mathbf{a}(t) \times \mathbf{b}t) = a(t) \times \frac{dbfb}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b}$$

5. if $t = t(u)$ $\dfrac{d}{du}(a(t)) = \dfrac{dt}{du} \cdot \dfrac{d\mathbf{a}}{dt}$

Since $\dfrac{d\mathbf{a}}{dt}$ is also a vector function we can define its derivative $\dfrac{d^2\mathbf{a}}{dt^2}$ etc.
The proof of one of these results will be given as an example. To differentiate $a(t) \times b(t)$

$$\frac{a(t+h) \times b(t+h) - a(t) \times b(t)}{\delta t}$$
$$= \frac{a(t+h) \times (b(t+h) - b(t))}{\delta t} + \frac{(a(t+h) - a(t))}{\delta t} \times b(t)$$
$$\to \mathbf{a}(t) \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b}(t) \quad \text{as } h \to 0$$

<u>Example</u>

Find $\dfrac{d}{dt}|\mathbf{a}(t)|, \quad \dfrac{d}{dt}|\mathbf{a}(t)|^2$

We do the second one first, for:

$|\mathbf{a}(t)|^2 = \mathbf{a}(t) \cdot \mathbf{a}(t)$

So $\dfrac{d}{dt}|\mathbf{a}(t)|^2 = \mathbf{a} \cdot \dfrac{d\mathbf{a}}{dt} + \dfrac{d\mathbf{a}}{dt} \cdot \mathbf{a} = 2\mathbf{a} \cdot \dfrac{d\mathbf{a}}{dt}$

Now $|a| = (a \cdot a)^{\frac{1}{2}}$

So $\dfrac{d}{dt}(a \cdot a)^{\frac{1}{2}} = \dfrac{1}{2}(a \cdot a)^{-\frac{1}{2}}\dfrac{d}{dt}(a \cdot a) = \dfrac{\mathbf{a} \cdot \frac{d\mathbf{a}}{dt}}{|a|} = \hat{\mathbf{a}} \cdot \dfrac{d\mathbf{a}}{dt}$

Notice that if $|a|$=constant this implies

$\dfrac{d}{dt}|a| = 0$ but not $\dfrac{d\mathbf{a}}{dt} = 0$

For $\hat{\mathbf{a}} \cdot \dfrac{d\mathbf{a}}{dt} = 0$ implies $\dfrac{d\mathbf{a}}{dt} = 0$ <u>OR</u> $\dfrac{d\mathbf{a}}{dt}$ is perpendicular to $\hat{\mathbf{a}}$

i.e. $\dfrac{d\mathbf{a}}{dt}$ is perpendicular to $\mathbf{a}$ if $\mathbf{a}$ is a vector of constant magnitude (e.g. a variable unit vector)

Geometrical Interpretation
Let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of a point on a curve in space described by means of the parameter t

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$\vec{PP'} = \mathbf{r}(t + \delta t) - \mathbf{r}(t)$
As $\delta t \to 0$. The direction of $\vec{PP'}$ tends towards that of the tangent vector at P.

$$\frac{d\mathbf{r}}{dt} = \lim_{h \to 0} \frac{\vec{PP'}}{\delta t}$$

so if this limit is nonzero then it is a vector whose direction is that of the tangent.
Now suppose that the parameter is s, the length of arc from one point A on the curve.

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The length of are PP' is approximately the same as the length of the chord PP'
So $|\vec{PP'}| \approx \delta s$
Thus $\left| \dfrac{\mathbf{r}(s + \delta s) - \mathbf{r}(s)}{\delta s} \right| \approx 1$
Thus as $\delta s \to 0$ we have $\dfrac{d\mathbf{r}}{ds} =$ unit tangent vector at P.

Example
Consider the curve $x = a \cos t \ y = a \sin t$

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Measure are length from A, $s=$ arc $AP = at$ so $t = \dfrac{s}{a}$
So in terms of the parameter s,
$\vec{OP} = \mathbf{r}(s) = a \cos \dfrac{s}{a}\mathbf{i} + a \sin \dfrac{s}{a}\mathbf{j}$
So $\dfrac{d\mathbf{r}}{ds} = -\sin \dfrac{s}{a}\mathbf{i} + \cos \dfrac{s}{a}\mathbf{j}$
Thus $\left| \dfrac{d\mathbf{r}}{ds} \right| = 1$

Also

$$\begin{aligned}\frac{d\mathbf{r}}{ds} &= \cos\left(\frac{\pi}{2}+\frac{s}{a}\right)\mathbf{i}+\sin\left(\frac{\pi}{2}+\frac{s}{a}\right)\mathbf{j}\\ &= \cos\left(\frac{\pi}{2}+t\right)\mathbf{i}+\sin\left(\frac{\pi}{2}+t\right)\mathbf{j}\end{aligned}$$

Thus $\frac{d\mathbf{r}}{ds}$ is a unit vector obtained from r by rotating $\mathbf{r}$ through $90°$. i.e. it is a unit vector in the direction of the tangent.
Since $\mathbf{r}(t) = a\cos t\mathbf{i} + a\cos t\mathbf{j}$
$\frac{d\mathbf{r}}{dt} = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$ so $\left|\frac{d\mathbf{r}}{dt}\right| = a$

However consider
$\mathbf{r}(t) = a\cos(t^2)\mathbf{i} + a\sin(t^2)\mathbf{j}$ - still on circle $|\mathbf{r}(t)| = a$ but
$\frac{d\mathbf{r}}{dt} = -2at\sin(t^2)\mathbf{i} + 2at\cos(t^2)\mathbf{j}$
So $\frac{d\mathbf{r}}{dt} \neq$constant, neither is $\left|\frac{d\mathbf{r}}{dt}\right|$ is a constant. In fact when $t = 0$, $\frac{d\mathbf{r}}{dt} = \mathbf{0}$
Example in $\mathbf{R}^3$
Let $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$
This represents a helix of radius a and pitch $2\pi b$

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$\frac{d\mathbf{r}}{dt} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}$
So $\frac{d\mathbf{r}}{dt}\cdot\mathbf{k} = b-$constant.
Thus the tangent vector makes a constant angle with the vertical.

Physical Interpretation
If we now consider t as a time parameter, and $\mathbf{r}(t)$ representing the position of a particle at time t than over a small interval of time, with the particle moving roughly in a straight line, the quotient

$$\frac{|r(t+\delta t)-r(t)|}{\delta t} \approx \frac{\text{distance traveled}}{\text{time taken}}$$

measures the average speed of the particle over the time interval from t to $t+\delta t$. We then define

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t\to 0}\frac{\mathbf{r}(t+\delta t)-\mathbf{r}(t)}{\delta t}$$

to be the instantaneous velocity at time t, when $\mathbf{r}(t)$ is differentiable. Notice that this nay not always exist, as when an impulse is applied, giving an instantaneous change in speed and / or direction.

It is the relationship between velocity and derivative which enabled Newton to solve so many dynamical problems when he invented the differential calculus.
In dealing with velocities there is a traditional notation, stemming from Newton.
We use $\dot{\mathbf{r}}$ to stand for $\dfrac{d\mathbf{r}}{dt}$
The derivative of velocity is called acceleration, $\ddot{\mathbf{r}}$. Another traditional notation is to use r for $|\mathbf{r}|$ (as it is easier to read in books)
Notice the distance between $\dot{\mathbf{r}} = \dfrac{d\mathbf{r}}{dt}$ and $\dot{r} = \dfrac{d|\mathbf{r}|}{dt}$
<u>Example: Motion in a circle</u>
Suppose a particle is moving round a circle, radius a, with a uniform speed, and it takes k seconds to perform one revolution. So in k seconds it travels and are of $2\pi a$. In t seconds it therefore travles an are of $\frac{2\pi a}{k}t$. We would expect its speed to be $\frac{2\pi a}{k}$.

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At time t
Arc $AP = a\theta = \dfrac{2\pi a}{k}t$
So $\theta = \dfrac{2\pi}{k}t$
Thus $\mathbf{r}(t) = a\cos\dfrac{2\pi}{k}t\mathbf{i} + b\sin\dfrac{2\pi}{k}t\mathbf{j}$
So

$$\begin{aligned}\dot{\mathbf{r}} &= -\frac{2\pi a}{k}\sin\frac{2\pi}{k}t\mathbf{i} + \frac{2\pi a}{k}\cos\frac{2\pi}{k}t\mathbf{j} \\ &= \frac{2\pi a}{k}\left(\cos\left(\theta + \frac{\pi}{2}\right)\mathbf{i} + \sin\left(\theta + \frac{\pi}{2}\right)\mathbf{j}\right)\end{aligned}$$

We see therefore that $|\dot{\mathbf{r}}| = \dfrac{2\pi a}{k}$ =constant $= v$
(v is a letter often used for speed)
and the direction of $\dot{\mathbf{r}}$ is that the tangent at P.

$$\begin{aligned}\ddot{\mathbf{r}} &= -\frac{4\pi^2 a}{k^2}\cos\frac{2\pi}{k}t\mathbf{i} - \frac{4\pi^2 a}{k}\sin\frac{2\pi}{k}t\mathbf{j} \\ &= \left(\frac{2\pi a}{k}\right)^2 \cdot \frac{1}{a}(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j})\end{aligned}$$

so the magnitude $|\ddot{\mathbf{r}}| = \dfrac{v^2}{a}$ and the direction of $\ddot{\mathbf{r}}$ is towards the centre from P.

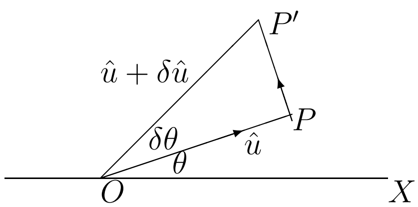
$$\ddot{\mathbf{r}} = -\frac{v^2}{a}\hat{\mathbf{r}}$$

Rotating unit vectors

We have already seen that if $\mathbf{a}(t)$ is a vector with $|\mathbf{a}|$ constant then $\mathbf{a}$ is perpendicular $\dfrac{d\mathbf{a}}{dt}$. I now want to analyse this situation a bit further.

Let $\hat{u}(t)$ be a unit vector.

Fix an origin O and let $\hat{u}(t)$ be the position vector od a point relative to O.



Let $X\hat{O}P = \theta$ the angle measured relative to some initial line OX through O.

Then $\vec{PP'} = \vec{OP'} - \vec{OP} = \delta\hat{u}$

So $\dfrac{\delta\hat{u}}{\delta t} = \dfrac{\vec{PP'}}{\delta t}$

Since $|\vec{OP}| = |\vec{OP'}| = 1$

So $\dfrac{|\vec{PP'}|}{\delta t} = \dfrac{2\sin\frac{1}{2}\delta\theta}{\delta t} = \dfrac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\cdot\dfrac{\delta\theta}{\delta t}$

Thus if $\hat{p}(t)$ is the vector perpendicular to $\hat{u}(t)$ in the direction of θ increasing we have

$$\frac{d\hat{u}}{dt} = \frac{d\theta}{dt}\hat{p} \quad \text{or} \quad \dot{\hat{u}} = \dot{\theta}\hat{p}$$

Radial and transverse components velocity and acceleration

Suppose a particle P is moving in some path, described in polar co-ordinates, so that the co-ordinates of P at time t are $(r(t), \theta(t))$

At the point P we wish to consider the components of velocity and acceleration in the direction of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$

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Notice that as P moves along the curve the directions of $\hat{r}$ and $\hat{\theta}$ change. They are rotating unit vectors.
Now $\mathbf{r} = \vec{OP} = r\hat{r}$ (remember $r = |\mathbf{r}|$)
So the velocity is given by:

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

So the radial component of velocity is $\dot{r}$ and the transverse component is $r\dot{\theta}$
To find the acceleration we differentiate again

$$\begin{aligned}
\ddot{\mathbf{r}} &= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\cdot\dot{\theta}\hat{r} \\
&= (\ddot{r} - r(\dot{\theta})^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}
\end{aligned}$$

So the radial component acceleration is $\ddot{r} - r\dot{\theta}^2$ The transverse component of acceleration is $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

Angular velocity
$\dot{\theta}$ defines the angular speed of a particle. Suppose a particle is moving in a circle centre O, so that r is constant.
Then $\frac{d\mathbf{r}}{dt} = r\dot{\theta}\hat{\theta}$
Let $\hat{n} = \hat{r} \times \hat{\theta}$ and let $\boldsymbol{\omega} = \dot{\theta}\hat{n}$
Then

$$\begin{aligned}
\boldsymbol{\omega} \times \mathbf{r} &= r\dot{\theta}(\hat{\mathbf{r}} \times \hat{\theta}) \times \hat{r} \\
&= r\dot{\theta}((\hat{r}\cdot\hat{r})\hat{\theta} - (\hat{r}\cdot\hat{\theta})\hat{r}) \\
&= r\dot{\theta}\hat{\theta}
\end{aligned}$$

So

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times r$$

$\boldsymbol{\omega}$ is called the angular velocity vector.
In general if r os not constant

$$\frac{d\mathbf{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \dot{r}\hat{r} + \boldsymbol{\omega} \times r$$

<u>Example</u>

A particle moves along the equiangular spiral $r = e^\theta$ with constant angular velocity about the origin. Prove that the acceleration is at right angles to the radius vector and proportional to its length.
$\boldsymbol{\omega} = \dot{\theta}\hat{n}$ is constant. So $\dot{\theta}$ is constant.
$r = e^\theta$ so $\dot{r} = e^\theta\dot{\theta} = r\dot{\theta}$
$\ddot{r} = \dot{r}\dot{\theta} = r(\dot{\theta})^2$
So the acceleration is given by

$$\begin{aligned} \ddot{r} &= (\ddot{r} - r(\dot{\theta})^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta}\hat{\theta}) \\ &= (r\dot{\theta}^2 - r\dot{\theta}^2)\hat{r} + 2r\dot{\theta}^2\hat{\theta} \\ &= 2r(\dot{\theta})^2\hat{\theta} \end{aligned}$$

So $\ddot{r}$ is perpendicular $\hat{\mathbf{r}}$ and $|\ddot{\mathbf{r}}| = r(\dot{\theta})^2$ or r as $\dot{\theta}$ is constant.

Tangential and normal components of velocity and acceleration
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s is the path length measured from a fixed point on the curve

$$\begin{aligned} \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} &= \dot{s}\hat{t} \quad \left(\frac{d\mathbf{r}}{ds} = \hat{t} \text{ already established}\right) \\ &= v\hat{t} \quad (v \text{ is speed}) \end{aligned}$$

It makes physical sense to say that the velocity is the rate at which the particle is traveling along the path, and that the direction of travel at any instance is equal to the tangential direction at that point.
Differentiating again gives

$$\ddot{r} = \ddot{s}\hat{t} + \dot{s}\frac{d\hat{t}}{dt}$$

t is a rotating unit vector, that angle of rotation being measured by the angle ϕ. So $\frac{d\hat{t}}{dt} = \dot{\phi}\hat{n}$
Thus $\ddot{\mathbf{r}} = \ddot{s}\hat{t} + \dot{s}\dot{\phi}\hat{n}$
Now $\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{ds} \cdot \frac{ds}{dt}$
$\frac{d\phi}{ds}$ measures how fast the angle ϕ is changing as we move along the curve. It gives a measure of how rapidly the curve is turning so we call it the curvature denoted by κ
<u>Example</u>
Consider a circle

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$\mathbf{r} = a\hat{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j}$
Measure are length from A. Then $s(t) = at$.

$$\frac{d\psi}{ds} = \frac{d\psi}{dt} \cdot \frac{dt}{ds} = \frac{\left(\dfrac{d\phi}{dt}\right)}{\left(\dfrac{ds}{dt}\right)} = \frac{1}{a}$$

So curvature $= \dfrac{1}{\text{radius}}$
For a general curve we call $\dfrac{ds}{d\psi} = \dfrac{1}{\frac{d\psi}{ds}}$ the radius of curvature ρ.
So we have $\dot{\psi} = \dfrac{\dot{s}}{\rho}$
Thus $\ddot{\mathbf{r}} = \ddot{s}\hat{r} + \dfrac{\dot{s}^2}{\rho}\hat{n}$ or $\ddot{\mathbf{r}} = \ddot{s}t + \rho(\dot{\psi})^2\hat{n}$
It follows immediatly that if a particle is moving at unifrom speed along a curve, $\ddot{s} = 0$ and so the acceleration is normal to the curve.
Also if the particle moves so that $\dot{\psi} =$ constant the the normal component is proportional to the radius of curvature.

Example
A particle moves on the curve

$$y = \log\sec x$$

in such a way that the tangent to the curve at the point where the particle is rotates at a uniform rate.
i.e. $\dot{\psi} = \text{constant} = k$.

Now $y = \log\sec x = -\log\cos x$ and so $\dot{y} = \tan x\dot{x}$ Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
So the velocity id given by
$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{x}\tan x\mathbf{j}$
The speed is therefore

$$|\dot{r}| = |\dot{x}|\sqrt{1 + \tan^2 x} = |\dot{x}|\sec x = v$$

The unit tangent vector to the path is given by $\hat{t} = \dfrac{\dot{V}}{v} = \cos x\mathbf{i} + \sin x\mathbf{j}$
Now $\dfrac{d\hat{t}}{dt} = \dot{\psi}\hat{n}$

So $\dot{\psi} = \left|\frac{d\hat{t}}{dt}\right| = |-sinx\dot{x}\mathbf{i} + \cos x\dot{x}\mathbf{j}| = |\dot{x}|$
Thus $|\dot{x}| = k$ so $\dot{x}$ is a constant.
The acceleration is given by

$$\ddot{\mathbf{r}} = \dot{x}\sec^2 x\dot{x}\mathbf{j} = \dot{x}^2\sec^2 x\mathbf{j} = v^2\mathbf{j}$$

as $\dot{x}$ is constant.
So the acceleration is in the direction parallel to the y-axis.
Now $v = \dot{s} = \frac{ds}{d\psi}\dot{\psi} = \rho k$
So $v^2 = k^2\rho^2$
Thus the acceleration is proportional to ρ^2
Motion with a rotating frame.
Suppose we have a plane uniformly rotating with respects to a fixed plane (like a gramophone turntable). Fix a pair of axis in the rotating plane.

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so $\hat{p}$ and $\hat{q}$are rotating unit vectors, and $\dot{\theta}$ =constant= w
Support a particle is moving around in the rotating plane and that its position vector at time t is $r(t)$. We find the components of $r(t)$ with respects to $\hat{p}$ and $\hat{q}$, so

$$\dot{\mathbf{r}}(t) = \dot{P}(t)\hat{p} + Q(t)\hat{q}$$

The velocity is given by

$$\begin{aligned}
\dot{\mathbf{r}}(t) &= \dot{P}(t)\hat{p} + P(t)\frac{d\hat{p}}{dt} + \dot{Q}(t)\hat{q} + Q(t)\frac{d\hat{q}}{dt} \\
&= \dot{P}(t)\hat{p} + P(t)\cdot\omega\hat{q} + \dot{Q}(t)\hat{q} + Q(t)(-\omega\hat{p}) \\
&= (\dot{P} + \omega Q)\hat{p} + (\dot{Q} + \omega P)\hat{q}
\end{aligned}$$

It acceleration is then given by

$$\begin{aligned}
\ddot{\mathbf{r}}(t) &= (\ddot{P} - \omega\dot{Q})\hat{p} + (\dot{P} - \omega Q)\cdot\omega\hat{q} + (\ddot{Q} + \omega\dot{P})\hat{q} - (\dot{Q} + \omega P)\omega\hat{p} \\
&= (\ddot{P} - 2\omega\dot{Q} = \omega^2 P)\hat{p} + (\ddot{Q} + 2\omega\dot{P} - \omega^2 Q)\hat{q} \\
&= (\ddot{P}\hat{p} + \ddot{Q}\hat{q}) + 2\omega(-\dot{Q}\hat{p} + \dot{P}\hat{q}) - \omega^2(P\hat{p} + Q\hat{q})
\end{aligned}$$

So this consists of three terms

(i) $\ddot{P}\hat{p} + \ddot{Q}\hat{q}$ if the acceleration of the particle relative to the rotating system.

(ii) $2\omega(-\dot{Q}\hat{q} + \dot{P}\hat{q})$ is the acceleration of the particle due to its velocity with in the rotating system

(iii) $-\omega^2(P\hat{p} + Q\hat{q}) = -w^2\mathbf{r}$ is the acceleration towards the centre due to the rotation of the system itself.

Special cases

(i) if the system is not rotating so $\omega = 0$ then $\ddot{\mathbf{r}} = \ddot{P}\hat{p} + \ddot{Q}\hat{q}$ - the normal acceleration.

(ii) If the particle is stationary relative to the rotating system then P and Q are constant and $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$ - motion in a circle.

Example
An insect crawls outwards along the spoke of a bicycle wheel rotation with uniform angular velocity ω. The insect crawls with uniform speed u relative to the spoke.
Let $\hat{p}$ be the unit vector along the insect's spoke so $\dot{P} = u$, $Q = 0$ and $\dot{Q} = 0$
Thus the velocity is

$$\dot{\mathbf{r}} = u\hat{p} + \omega P\hat{q}$$

direction $\tan^{-1}\dfrac{\omega P}{u}$ to spoke and the acceleration is

$$\ddot{\mathbf{r}} = -\omega^2 P\hat{p} + 2\omega u\hat{q}$$

direction $\tan^{-1}\left(-\dfrac{2u}{\omega p}\right)$ to spoke

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Differential Geometry of curves in space
Suppose we have a curve in space described parametrically as

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

We have already seen that if $\dfrac{d\mathbf{r}}{dt} \neq \mathbf{0}$, then it is parallel to the tangent vector $\hat{t}$, and that if the parameter is the arc length s then $\dfrac{d\mathbf{r}}{ds} = \hat{t}$.
(Note: I have used u as the parameter rather then t to avoid confusion with $\hat{t}$)
Now since $\hat{t} \cdot \hat{t} = 0$ then $\dfrac{d}{ds}(\hat{t} \cdot \hat{t}) = 0$ So $\hat{t} \cdot \dfrac{d\hat{t}}{ds} = 0$ So $\dfrac{d\hat{t}}{ds}$ is perpendicular to t.
Then we write $\dfrac{d\hat{t}}{ds} = \kappa\hat{n}$ with $\kappa > 0$.

The magnitude of $\frac{d\hat{t}}{ds}(=\kappa)$ measures how fast the tangent is turning with respects to the are length and it is called **curvature**. $\hat{n}$ is called the **principle unit normal**.

We then choose $\hat{\mathbf{b}}$ so that $\hat{t}$, $\hat{n}$, $\hat{b}$. form a right-handed system at the point on the curve.
i.e. $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$
b is called the **binormal vector**
Now

$$\begin{aligned} \frac{d\hat{b}}{ds} &= \frac{d}{ds}(\hat{t} \times \hat{n}) \\ &= \frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \\ &= \hat{t} \times \frac{d\hat{n}}{ds} \end{aligned}$$

So $\frac{d\hat{b}}{ds}$ is perpendicular to $\hat{t}$. Also $\frac{d\hat{b}}{ds}$ is perpendicular to $\hat{b}$ as $\hat{b}$ is a unit vector. Thus $\frac{d\hat{b}}{ds}$ is parallel to $\hat{n}$ (or $-\hat{n}$). $\frac{d\hat{b}}{ds}$ measures the rate at which $\hat{b}$ rotates with respect to are length. It measures the amount of twist or torsion. We choose the torsion so that it is positive if the rotation about $\hat{t}$ is right handed as s increases, and negative if it is left handed. This means that we need to write

$$\frac{d\hat{b}}{ds} = -\tau\hat{n}$$

Now $\hat{n} = \hat{t} \times \hat{t}$ so

$$\begin{aligned} \frac{d\hat{n}}{ds} &= \frac{d\hat{b}}{ds} \times \hat{t} + \hat{b} \times \frac{d\hat{t}}{ds} \\ &= -\tau\hat{n} \times \hat{t} + \hat{b} \times \kappa\hat{n} \\ &= \tau\hat{g} - \kappa\hat{t} \end{aligned}$$

The three formulae we have found

$$\frac{d\hat{t}}{ds} = \kappa\hat{n} \qquad \frac{d\hat{b}}{ds} = -\tau\hat{n} \qquad \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}$$

are known as the Serret-Frenet formulae.

Example

Consider the helix

$$\mathbf{r} = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$$

Clearly from this formula we can calculate derivatives with respect to u. So to calculate derivatives with respect to s we shall need to know $\dfrac{du}{ds}$.

$\dfrac{d\mathbf{r}}{du} = -\sin u\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$

The unit tangent vector is $\dfrac{d\mathbf{r}}{ds}$

S $\hat{t} = \dfrac{d\mathbf{r}}{ds} = \dfrac{\left(\dfrac{d\mathbf{r}}{du}\right)}{\left|\frac{d\mathbf{r}}{du}\right|} = \dfrac{1}{\sqrt{2}}(-\sin u\mathbf{i} + \cos u\mathbf{j} + \mathbf{k})$

But $\dfrac{d\mathbf{r}}{ds} = \dfrac{d\mathbf{r}}{du} \cdot \dfrac{du}{ds}$ So $\dfrac{du}{ds} = \dfrac{1}{\sqrt{2}}$

Now $\kappa\hat{n} = \dfrac{d\hat{\mathbf{t}}}{ds} = \dfrac{d\hat{\mathbf{t}}}{du} \cdot \dfrac{du}{ds} = \dfrac{1}{2}(-\cos u\mathbf{i} - \sin u\mathbf{j})$

$\kappa = \left|\dfrac{d\hat{\mathbf{t}}}{ds}\right| = \dfrac{1}{2}$ (same at all points), so $\hat{n} = -\cos u\mathbf{i} - \sin u\mathbf{j}$

The principle normal is therefore parallel to the $\mathbf{i} - \mathbf{j}$ plane and the points are inwards towards the $\mathbf{k}$ axis.

Now $\hat{b} = \hat{t} \times \hat{n} = \dfrac{1}{\sqrt{2}}(\sin u\mathbf{i} - \cos u\mathbf{j} + \mathbf{k})$

and $-\tau\hat{n} = \dfrac{d\hat{\mathbf{b}}}{ds} = \dfrac{d\hat{\mathbf{b}}}{du} \cdot \dfrac{du}{ds} = \dfrac{1}{2}(\cos u\mathbf{i} + \sin \mathbf{j}) = -\dfrac{1}{2}\hat{n}$

So $\tau = \dfrac{1}{2}$ - same at all points.

Now $\dfrac{d\hat{\mathbf{n}}}{ds} = \dfrac{d\hat{\mathbf{n}}}{du} \cdot \dfrac{du}{ds} = \dfrac{1}{\sqrt{2}}(\sin u\mathbf{i} - \cos u\mathbf{j})$

$$\begin{aligned} \tau\hat{b} = \kappa\hat{t} \quad &= \frac{1}{2}\frac{1}{\sqrt{2}}(\sin u\mathbf{i} - \cos u\mathbf{j} + \mathbf{k}) - \frac{1}{2}\frac{1}{\sqrt{2}}(-\sin u\mathbf{i} + \cos u\mathbf{j} + \mathbf{k} \\ &= \frac{1}{\sqrt{2}}(\sin u\mathbf{i} - \cos u\mathbf{j}) \end{aligned}$$

Which verifies the third Serret Frenet formula in this case.

Now the plane containing the tangent and principle normal at a point (called the **osculating plane**) has the binormal as a normal vector.

So for the helix, at the point u, its equations is

$$(\sin u)x - (\cos u)y + z = k$$

$$k = (\sin u)\cos u - \cos u \sin u + u = u$$

So the equation is

$$(\sin u)x - (\cos u)y + z = u$$

e.g. at the point $u = \frac{\pi}{2}$ the equations is

$$x + z = \frac{1}{2}\pi$$

The plane containing the tangent and binoraml is called the **rectifying plane** and it has $\hat{n}$ as a normal vector. So for the helix at the point u, its equations is

$$(-\cos u)x + (-\sin u)y = (-\cos u)(\cos u) + (-\sin u)(\sin u) = -1$$

So the equation is

$$(\cos u)x + (\sin y) = 1$$

e.g. at the point $u = \frac{\pi}{2}$ the equation is

$$y = 1$$

The plane containing the normal and binormal is the **normal plane**, having $\hat{t}$ as a normal vector. It equation is

$$(-\sin u)x + (\cos u)y + z = (-\sin u)(\cos u) + (\cos u)(\sin u) + u$$

i.e.

$$(-\sin u)x + (\cos u)y + z = u$$