# From Set Theory to Predicate Logic

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February 16, 2022

## 1 Sets

### Definition: Set

A *set* is an unordered collection of objects, without duplicates. A set $A$ containing the objects $a$, $b$ and $c$ is written as $A = \{a, b, c\}$.

### Definition: Empty Set

The *empty set* (written $\emptyset$ or $\{\}$) is the set containing nothing.

### Definition: Set Membership

An object $a$ is a *member* of a set $A$ (written $a \in A$) if it is contained within that collection.

**Note:** $a \in A$ can be read as "$a$ is a member of $A$" or "$a$ belongs to $A$".

### Definition: Set Equality

Two sets which contain the same objects are considered to be equal (the order of the objects is unimportant).

**Example:** If $A = \{a, b, c\}$, $B = \{b, a, c\}$ and $C = \{a, b, d\}$, then $A = B$ but $A \neq C$

### Definition: Cardinality

The *cardinality* of a set $A$ (written $|A|$ or $\#A$) is the number of members of A.

**Example:** If $A = \{a, b, c\}$, then $|A| = 3$

### Definition: Subset

A set $A$ is a *subset* of a set $B$ (written $A \subseteq B$) if every member of $A$ is also a member of $B$.

**Example:** If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, c, d\}$, then $A \subseteq B$, but $A \nsubseteq C$

### Definition: Strict Subset

A set $A$ is a *strict subset* of a set $B$ (written $A \subset B$) if every member of $A$ is also a member of $B$, and $A \neq B$.

**Example:** If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, b\}$, then $A \subset B$, but $A \not\subset C$

### Definition: Set Intersection

The *intersection* of two sets $A$ and $B$ (written as $A \cap B$) is the set containing every object which is **both** a member of $A$ **and** a member of $B$.

**Example:** If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A \cap B = \{a, c\}$

### Definition: Set Union

The *union* of two sets $A$ and $B$ (written as $A \cup B$) is the set containing every object that is a member of $A$ **or** a member of $B$, **or** a member of both $A$ and $B$.

**Example:** If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A \cup B = \{a, b, c, d\}$

**Mnemonic:** $\cup$ stands for U(nion)

### Definition: Set Difference

The *difference* of two sets $A$ and $B$ (written as $A - B$) is the set of every object that is a member of $A$ but not a member of $B$.

**Example:** If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A - B = \{b\}$

**Note:** $(A - B) \neq (B - A)$

### Definition: Powerset

The *powerset* of $A$ (written $\mathbb{P}(A)$ or $2^A$) is the set containing all possible subsets of $A$, including $A$ and the empty set.

**Example:** If $A = \{a, b, c\}$, then

$$\mathbb{P}(A) = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \{\}\}$$

**Note:** $|\mathbb{P}(A)| = 2^{|A|}$.

## Definition: Set Comprehension

Rather than explicitly list the members of a set as $A = \{a_1, \ldots, a_n\}$ ,we can define a set by specifying the properties that its members must have. This is known as *set comprehension.*

Set comprehension is expressed using *set-builder notation*, for which the general form is $\{x : \phi(x)\}$, where $x$ is a variable and $\phi(x)$ is a predicate containing $x$ which holds true for all members of the set. $\{x : \phi(x)\}$ can be read as "the set of $x$ for which $\phi(x)$ is true".

**Example:** $\{x : x \in \mathbb{Z} \wedge x > 0\}$
The set of positive integers - $\mathbb{Z}$ is the set of integers.
Read as: "the set of $x$'s where $x$ is an integer and $x$ is greater than zero".

**Example:** $\{x : x \in \mathbb{Z} \wedge x = x^2\}$
The set of integers which are equal to their square: $\{0, 1\}$

**Example:** $\{\langle x, y\rangle : x \in A \wedge y \in B\}$
The set of pairs $\langle x, y\rangle$ where $x$ is a member of set $A$ and $y$ is a member of set $B$. This is the definition of the Cartesian product $A \times B$ using set-builder notation.

## Definition: Tuple

A *tuple* is an ordered collection of objects, which may include duplicates. The tuple containing $a$, $b$, $c$ and $a$, in that order, is written $\langle a, b, c, a\rangle$

## Definition: Arity

The *degree* or *arity* of a tuple is the number of objects in the tuple.

## Definition: Pair

A tuple containing two objects (a tuple of arity 2) is known as a *pair.*

## Definition: Cartesian Product

The *Cartesian product* of two sets $A$ and $B$ (written $A \times B$) is a set of pairs, where each pair contains one member from $A$ and one member from $B$, and which contains all possible combinations of members from $A$ and $B$.

**Example:** If $A = \{a, b, c\}$ and $B = \{c, d, e\}$, then

$$A \times B = \{\langle a, c\rangle, \langle a, d\rangle, \langle a, e\rangle, \langle b, c\rangle, \langle b, d\rangle, \langle b, e\rangle, \langle c, c\rangle, \langle c, d\rangle, \langle c, e\rangle\}$$

**Note:** $|A \times B| = |A| * |B|$

### Definition: Binary Relation

A *binary relation* $R$ from set $A$ to set $B$ is a set of pairs, where each pair contains one member from $A$ and one member from $B$.

**Example:** If $A = \{a, b, c\}$ and $B = \{c, d, e\}$, then a possible relation $R$ from $A$ to $B$ might be:

$$R = \{\langle a, c\rangle, \langle a, e\rangle, \langle b, d\rangle, \langle c, c\rangle, \langle c, d\rangle, \}$$

**Note:** $R \subseteq A \times B$

### Definition: Domain

The *domain* of a binary relation $R$ is the set that the relation goes *from*.

**Example:** The domain of $R$ in the above example is $A$.

### Definition: Range

The *range* of a binary relation $R$ is the set that the relation goes *to*.

**Example:** The range of $R$ in the above example is $B$.

**Mnemonic:** the range of a cannon is the distance **to** which it can fire a cannonball.

# 2 Logic

## 2.1 Propositional Calculus

### Definition: Truth Values

The truth values are *true* and *false* (sometimes written as $\top$ and $\bot$ respectively).

### Definition: Proposition

A *proposition* is a truth-valued expression. That is, a proposition can either be *true* or *false*.

**Example:** "$a \in A$" is a proposition (either $a$ is a member of $A$, in which case "$a \in A$" is *true*, or $a$ is not a member of $A$, in which case "$a \in A$" is *false*). "$A \times B$" is not a proposition, because its value is a set.

### Definition: Logical Connectives

Propositions may be combined to form *compound propositions* by using the *logical connectives*: negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$) and implication ($\Rightarrow$) .

### Definition: Negation

The *negation* of a proposition $\phi$ (written as $\neg\phi$, and read as "not $\phi$") is *true* if $\phi$ is *false*.

| $\phi$ | $\neg\phi$ |
|---|---|
| false | true |
| true | false |

### Definition: Conjunction (logical and)

The *conjunction* of two propositions $\phi$ and $\psi$ (written as $\phi \wedge \psi$, and read as "$\phi$ and $\psi$") is *true* if both $\phi$ **and** $\psi$ are *true*.

**Mnemonic:** $\wedge$ stands for "And"

| $\phi$ | $\psi$ | $\phi \wedge \psi$ |
|---|---|---|
| false | false | false |
| false | true | false |
| true | false | false |
| true | true | true |

### Definition: Disjunction (logical or)

The *disjunction* of two propositions $\phi$ and $\psi$ (written as $\phi \vee \psi$, and read as "$\phi$ or $\psi$") is *true* if either $\phi$ is *true* **or** $\psi$ is *true* (**or** if both $\phi$ and $\psi$ are *true* – $\vee$ is the inclusive-or).

**Mnemonic:** $\vee$ stands for "vel" (the Latin for "or")

| $\phi$ | $\psi$ | $\phi \vee \psi$ |
|---|---|---|
| false | false | false |
| false | true | true |
| true | false | true |
| true | true | true |

### Definition: Implication

A *material implication* $\phi \Rightarrow \psi$ (read as "$\phi$ implies $\psi$" or "if $\phi$ then $\psi$") is a fourth common logical operator; it is *true* unless $\phi$ is *true* and $\psi$ is *false*.

| $\phi$ | $\psi$ | $\phi \Rightarrow \psi$ |
|---|---|---|
| false | false | true |
| false | true | true |
| true | false | false |
| true | true | true |

#### Definition: Equivalences

The following equivalences hold between expressions in propositional logic:

$\neg\neg\phi \equiv \phi$
$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$
$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$
$\phi \Rightarrow \psi \equiv \neg\phi \vee \psi$

### 2.2 Predicate Logic

Predicate logic extends propositional logic by adding *variables*, *functions*, *predicates* and *quantification* to the connectives described above. Predicate logic distinguishes between *terms* and *formulas*.

#### Definition: Term

A term is a predicate logic expression which is not truth-valued.

#### Definition: Formula

A *formula* is a predicate logic expression which is truth-valued.

#### Definition: Variables

Variables are indicated with lower case letters: $x, y, z$ etc. All variables are terms.

#### Definition: Function

A *function* is an expression $f(t_1, t_2, \ldots t_n)$ where each $t_i$ is a term and $f$ is a function symbol with a valence of $n$ (i.e. it takes $n$ arguments). A function with a valence of 0 is considered to be a constant term.

**Example:** In the expression $john$, $john$ is a constant (a function with a valence of 0).

**Example:** In the expression $fatherOf(john)$, $fatherOf$ is a function of valence 1.

#### Definition: Predicate

A *predicate* is a truth-valued expression $p(t_1, t_2, \ldots t_n)$ where each $t_i$ is a term and $p$ is a function symbol with a valence of $n$ (i.e. it takes $n$ arguments).

**Example:** In the expression $likes(john, mary)$, $likes$ is a predicate with a valence of 2.

**Definition: Quantification**

There are two quantifiers: $\exists$ and $\forall$, respectively read as "there exists" and "for all". A quantified formula consists of a quantifier, a variable and a formula.

If $\phi$ is a formula and $x$ is a variable, then $\exists x.\phi$ can be read as "there exists some $x$ such that $\phi$ is true".

If $\phi$ is a formula and $x$ is a variable, then $\forall x.\phi$ can be read as "for all $x$, $\phi$ is true".

**Example:** $\forall x.man(x) \Rightarrow mortal(x)$ (i.e. for all x, if man(x) is true then mortal(x) is true - or in other words, "all men are mortal")