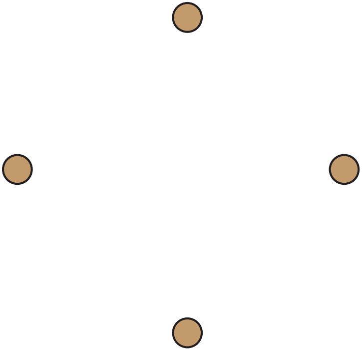
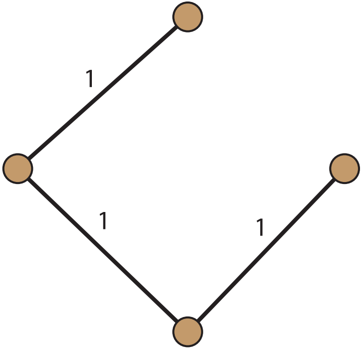
Large scale structure of metric spaces

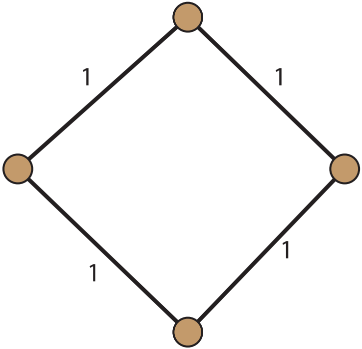
Jacek Brodzki

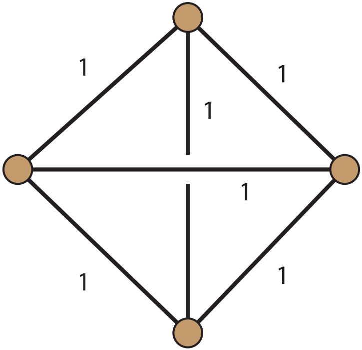
University of Southampton

Simple shapes

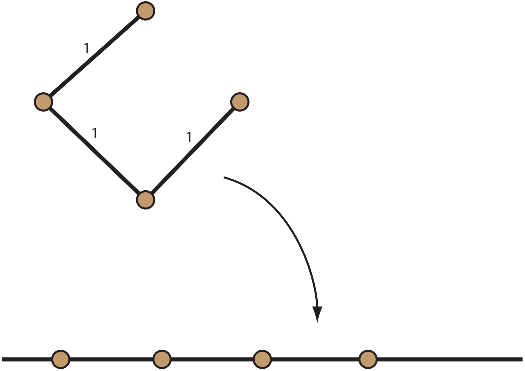



1
1
1

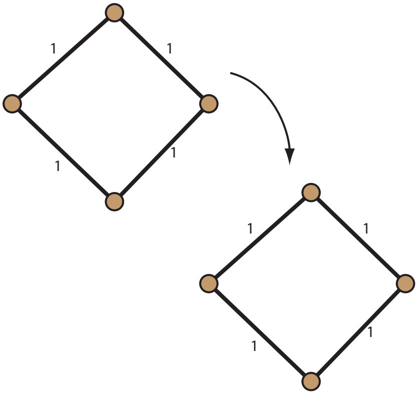

1
1
1
1


1
1
1
1
1
1

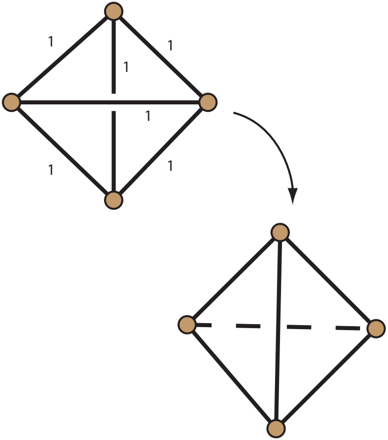
How many dimensions?



How many dimensions?



How many dimensions?



Metric spaces

Definition

Let X be a non-empty set. A *metric* (or a distance function) on X is a map $d : X \times X \to \mathbb{R}$ which satisfied the following properties:

1. d is *positive definite*: for every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. d is *symmetric*: for every $x, y \in X$, $d(x, y) = d(y, x)$.
3. d satisfies the *triangle inequality*: for every $x, y, z \in X$

$$d(x, z) \leq d(x, y) + d(y, z)$$

Examples of metrics on $\mathbb{R}^n$

The *Euclidean metric* For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ we define

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

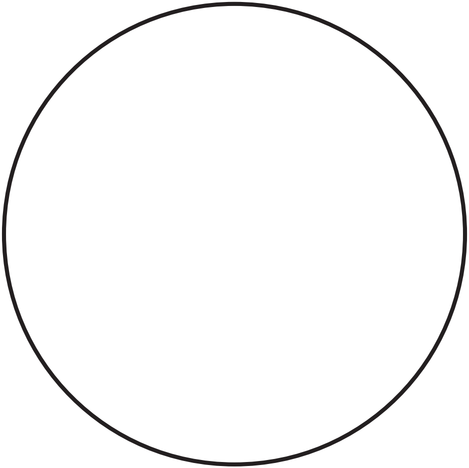
The *taxi-cab metric, or the ℓ^1-metric*:

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n + y_n|$$

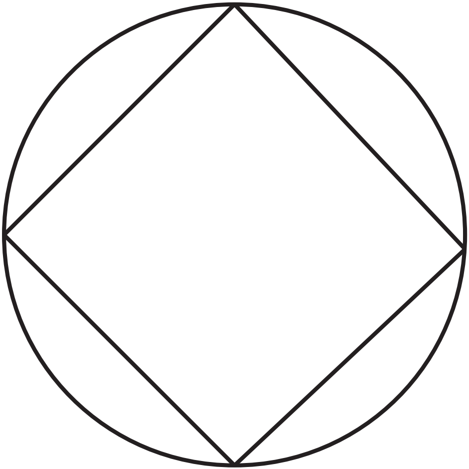
The *supremum metric*:

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n + y_n|\}$$

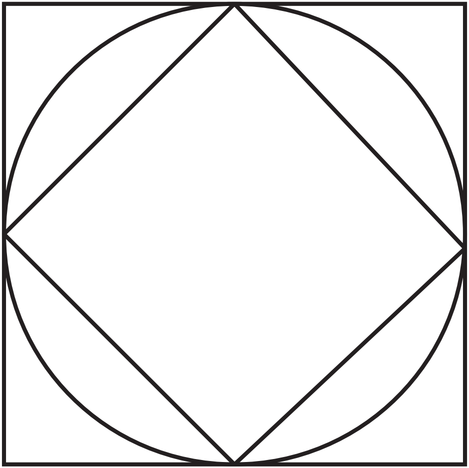
Metric determines shape



Metric determines shape



Metric determines shape



Analysis on sets

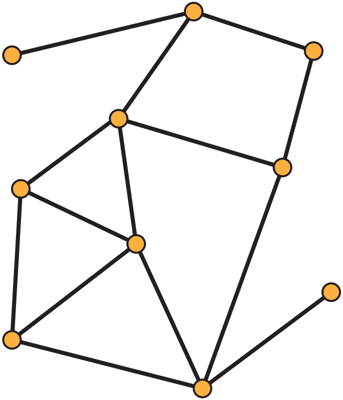
- Let X be a countable set. A Hilbert space canonically associated with X:

$$\ell^2(X) = \left\{ f : X \to \mathbb{C} \mid \sum_{x \in X} |f(x)|^2 < \infty \right\}$$

- Canonical orthonormal basis: $\{\delta_x\}$, $f = \sum_{x \in X} f_x \delta_x$, $f_x \in \mathbb{C}$.
- Transformations of X give rise to operators on $\ell^2(X)$, e.g., a bijection $\phi : X \to X$ becomes a unitary operator

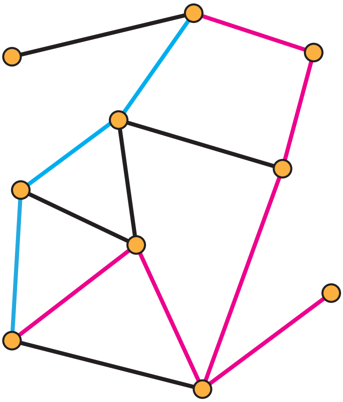
$$U_\phi : \sum f_x \delta_x \mapsto \sum f_x \delta_{\phi(x)}$$

Graphs provide natural examples of discrete metric spaces:



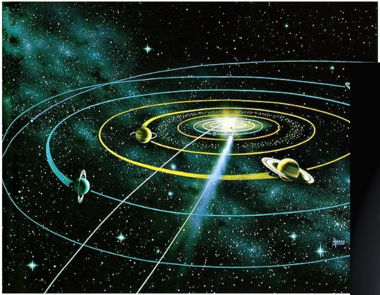
Path metric

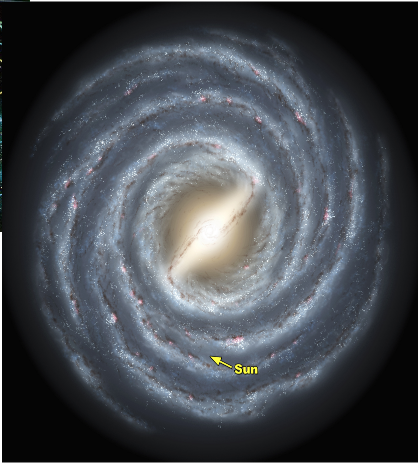
In a graph, it is natural to define a metric between points to be the length of the *shortest* path between them:



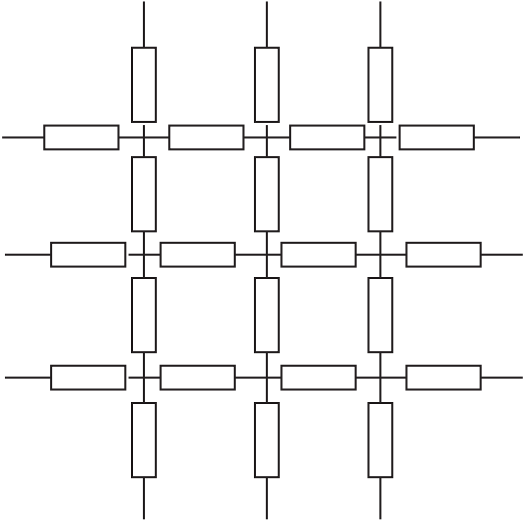
Large scale view

- There is no structure theory for discrete metric spaces;
- Key features of a space can be determined by studying it from a ‘large distance’



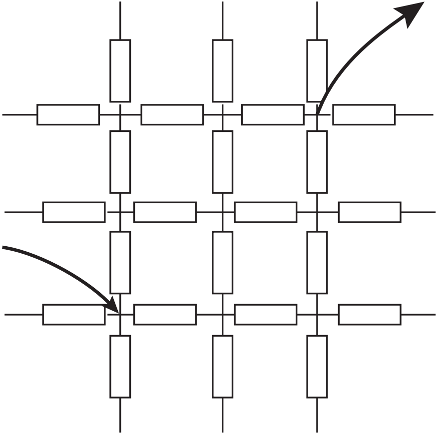

Sun

Metrics and function: Network of resistors



Metrics and function

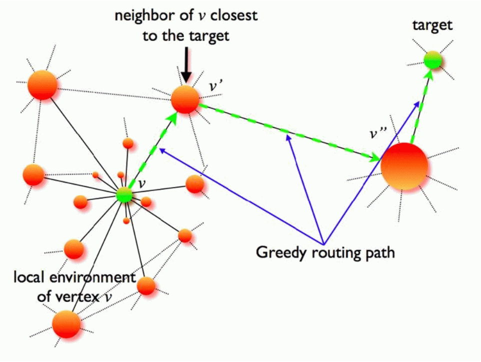
A distance between two points can be defined by measuring voltage drop resulting from passing 1 amp of current between them.



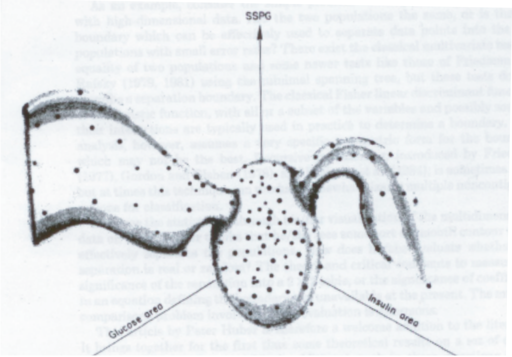
Greedy routing

The problem of finding the most efficient route between two points depends on the function of the network.

Picture from physorg.com

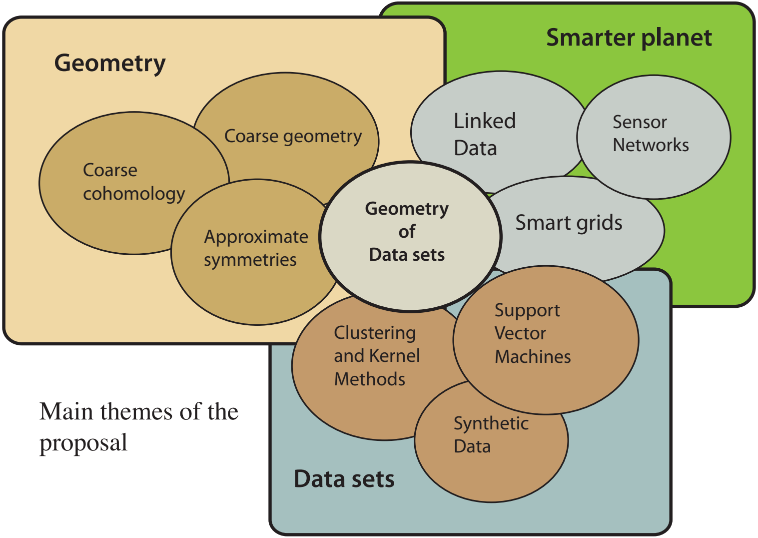


Topology of data



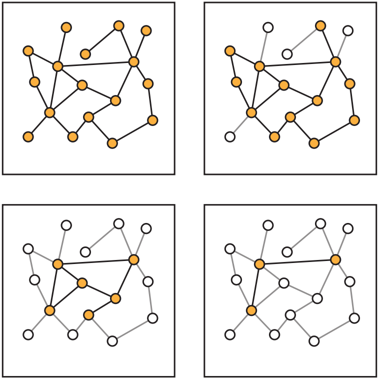
From: *Annals of Statistics*, Vol. 13, No. 2 June, 1985

Mathematics for digital economy



Main themes of the proposal

The essence of the topological approach is to find the essential core of the system.



Subgraphs consisting of vertices of valency at least: 1,2,3,4.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\phi : X \to Y$ is called *distance-preserving* if, and only if, $d_Y(\phi(x), \phi(y)) = d_X(x, y)$ for all $x, y \in X$.
An *isometry* is a distance-preserving *bijection* between two metric spaces.

Example

$\phi : \mathbb{R}^2 \to \mathbb{C}$ by $(a, b) \mapsto a + bi$. This is an isometry if $\mathbb{R}^2$ is equipped with the euclidean metric.

Coarse maps

Definition

A map $f : X \to Y$ of metric spaces is *coarse* if there exist two functions $\rho_\pm : \mathbb{R} \to \mathbb{R}$, $\rho_\pm(r) \to \infty$ as $r \to \infty$ such that for all $x, y \in X$

$$\rho_-(d_X(x,y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x,y))$$

Coarse maps have a controlled amount of distortion. Maps into spaces of known geometry (e.g., Hilbert spaces) are particularly useful.
The three metrics d_∞, d_1, d_2 on $\mathbb{R}^n$ are coarsely equivalent but not isometric.