Mathematical formalism: budgets, feedbacks and linear stability

By Kevin Oliver. Version 1.2, February 2010, for SOES6006

1 Budgets

A budget is a quantification of the net fluxes of a property into and out of a reservoir, which must be able to explain the net change in that property in the reservoir (e.g. heat in the atmosphere/ocean/ Earth). In general,

$$\frac{\partial}{\partial t} \int C \mathrm{d}V = \Sigma_{i=1}^{n} F_i^C , \qquad (1)$$

where C is a conservative property of interest, and we are considering fluxes of C (F^C) into the reservoir. (fluxes out of the reservoir can be represented as negative fluxes into the reservoir). Note that units of F_i^C are the units of C multiplied by the units of volume flux (m^3 s^{-1}) This equation can be easier to use if rewritten:

$$V\dot{C} = F_{in}^C - F_{out}^C , \qquad (2)$$

where C is now the mean property value in the reservoir, and we have used:

$$\dot{C} = \frac{\partial C}{\partial t},$$
$$F_{in}^C = \Sigma_{i=1}^{n} F_i^C , \qquad F_i^C > 0$$
$$F_{out}^C = -\Sigma_{i=1}^{n} F_i^C , \qquad F_i^C < 0$$

and assumed that the volume of the reservoir does not change in time. Note that F_{in} and F_{out} can include creation or destruction of the property within the reservoir (e.g. radioactive decay).

2 Positive and negative feedbacks

A system has feedbacks if any flux of property C into the reservoir, F_i^c, is a function of C (this includes fluxes out of the system and extends to internal generation terms.) The responce of F_i^c to C need not be immediate; there is feedback even if C only affects F_i^c indirectly. If F_i^c increases as C increases, this is a positive feedback. If F_i^c decreases as C increases, this is a negative feedback. That is:

$$\frac{\partial F_i^C}{\partial C} < 0 \rightarrow \text{Negative feedback}$$
$$\frac{\partial F_i^C}{\partial C} > 0 \rightarrow \text{Positive feedback} \qquad (3)$$

Negative F_i^c was reinterpreted as a positive flux out of the system in Equation (2). In this view positive feedback exists when a flux of C into the reservoir increases as C increases, or when a flux of C out of the reservoir decreases as C increases. Similarly, negative feedback exits when a flux of C into the reservoir decreases as C increases, or when a flux of C out of the reservoir increases as C increases.

3 System equilibria

"Equilibrium" in system analysis means that the properties of a system are not changing in time (c.f. equilibrium in mechanics, meaning an object's velocity is not changing). This means that any term containing a time derivative must be zero. For the generalised system with a conservative quantity, give in Equation (2), the criterion for equilibrium is:

$$F_{in}^C = F_{out}^C . \qquad (4)$$

The fluxes out of the reservoir must equal the fluxes into the reservoir.

4 Linear stability of an equilibrium in a one variable system

The Lyapunov test for linear stability of an equilibrium in a system is that *a system, if perturbed from equilibrium by a small amount, will return to equilibrium.* For a one-variable system, this is equivalent to stating that negative feedbacks must outweigh positive feedbacks at equilibrium, in order for the equilibrium to be stable. This criterion may be expressed:

$$\begin{aligned}
&\frac{\partial(F^C_{in} - F^C_{out})}{\partial C} < 0, \Rightarrow \text{Stable equilibrium} \\
&\frac{\partial(F^C_{in} - F^C_{out})}{\partial C} = 0, \Rightarrow \text{Critically stable equilibrium} \\
&\frac{\partial(F^C_{in} - F^C_{out})}{\partial C} > 0, \Rightarrow \text{Unstable equilibrium (also metastable equilibrium)}
\end{aligned} \tag{5}$$

However, we have already seen that, at equilibrium, $F^C_{in} - F^C_{out} = 0$ (see Eqn 4). Together with Eqn (5), this means that in a stable system, a positive perturbation in C must lead to a negative value of $F^C_{in} - F^C_{out}$ (similarly a negative perturbation in C must lead to a positive value of $F^C_{in} - F^C_{out}$).

5 Evolution of a one-variable system perturbed from equilibrium

The evolution of a one-variable system perturbed from equilibrium by a small amount can, in general, be written

$$\dot{C} = kC', \tag{6}$$

If we rewrite this

$$\frac{\partial C'}{\partial t} = kC',$$

it is straightforward to see that the equation can be rearranged:

$$\int \frac{1}{C'} \mathrm{d}C' = \int k \mathrm{d}t$$

and integrated:

$$\ln(C') + \ln(C_0^{-1}) = kt,$$

where $\ln(C_0^{-1})$ is an arbitrary constant of integration. This can be rearranged:

$$\ln(C_0^{-1}C') = kt,$$

and finally

$$C' = C_0 e^{kt}. \tag{7}$$

We see that if the growth constant k is positive, we have exponential growth, and if k is negative, we have exponential decay. This confirms the result that negative feedbacks are required for an equilibrium to be stable. However, CARE IS NEEDED. Sometimes, even within this course, k (or λ or other symbol) is used as a decay constant rather than a growth constant ($\dot{C} = -kC'$ instead of $\dot{C} = kC'$), in which case positive values would indicate decay and negative values would indicate growth. However, it is always possible to tell how the term is used from the equation and/or physical description of the system.

k (dimension of time^{-1}) gives the rate at which perturbations decay or grow; $1/k$ is the e-folding timescale.

Copyright statement