

Mathematical formalism: budgets, feedbacks and linear stability

By Kevin Oliver. Version 1.2, February 2010, for SOES6006

1 Budgets

A budget is a quantification of the net fluxes of a property into and out of a reservoir, which must be able to explain the net change in that property in the reservoir (e.g. heat in the atmosphere/ocean/Earth). In general,

$$\frac{\partial}{\partial t} \int C dV = \sum_{i=1}^n F_i^C, \quad (1)$$

where C is a conservative property of interest, and we are considering fluxes of C (F^C) into the reservoir. (fluxes out of the reservoir can be represented as negative fluxes into the reservoir). Note that units of F_i^C are the units of C multiplied by the units of volume flux ($\text{m}^3 \text{s}^{-1}$). This equation can be easier to use if rewritten:

$$V \dot{C} = F_{in}^C - F_{out}^C, \quad (2)$$

where C is now the mean property value in the reservoir, and we have used:

$$\begin{aligned} \dot{C} &= \frac{\partial C}{\partial t}, \\ F_{in}^C &= \sum_{i=1}^n F_i^C, \quad F_i^C > 0 \\ F_{out}^C &= -\sum_{i=1}^n F_i^C, \quad F_i^C < 0 \end{aligned}$$

and assumed that the volume of the reservoir does not change in time. Note that F_{in} and F_{out} can include creation or destruction of the property within the reservoir (e.g. radioactive decay).

2 Positive and negative feedbacks

A system has feedbacks if any flux of property C into the reservoir, F_i^C , is a function of C (this includes fluxes out of the system and extends to internal generation terms.) The response of F_i^C to C need not be immediate; there is feedback even if C only affects F_i^C indirectly. If F_i^C increases as C increases, this is a positive feedback. If F_i^C decreases as C increases, this is a negative feedback. That is:

$$\begin{aligned} \frac{\partial F_i^C}{\partial C} < 0 &\rightarrow \text{Negative feedback} \\ \frac{\partial F_i^C}{\partial C} > 0 &\rightarrow \text{Positive feedback} \end{aligned} \quad (3)$$

Negative F_i^C was reinterpreted as a positive flux out of the system in Equation (2). In this view positive feedback exists when a flux of C into the reservoir increases as C increases, or when a flux of C out of the reservoir decreases as C increases. Similarly, negative feedback exists when a flux of C into the reservoir decreases as C increases, or when a flux of C out of the reservoir increases as C increases.

3 System equilibria

“Equilibrium” in system analysis means that the properties of a system are not changing in time (c.f. equilibrium in mechanics, meaning an object’s velocity is not changing). This means that any term containing a time derivative must be zero. For the generalised system with a conservative quantity, give in Equation (2), the criterion for equilibrium is:

$$F_{in}^C = F_{out}^C. \quad (4)$$

The fluxes out of the reservoir must equal the fluxes into the reservoir.

4 Linear stability of an equilibrium in a one variable system

The Lyapunov test for linear stability of an equilibrium in a system is that *a system, if perturbed from equilibrium by a small amount, will return to equilibrium*. For a one-variable system, this is equivalent to stating that negative feedbacks must outweigh positive feedbacks at equilibrium, in order for the equilibrium to be stable. This criterion may be expressed:

$$\begin{aligned}\frac{\partial(F_{in}^C - F_{out}^C)}{\partial C} < 0, &\Rightarrow \text{Stable equilibrium} \\ \frac{\partial(F_{in}^C - F_{out}^C)}{\partial C} = 0, &\Rightarrow \text{Critically stable equilibrium} \\ \frac{\partial(F_{in}^C - F_{out}^C)}{\partial C} > 0, &\Rightarrow \text{Unstable equilibrium (also metastable equilibrium)}\end{aligned}\tag{5}$$

However, we have already seen that, at equilibrium, $F_{in}^C - F_{out}^C = 0$ (see Eqn 4). Together with Eqn (5), this means that in a stable system, a positive perturbation in C must lead to a negative value of $F_{in}^C - F_{out}^C$ (similarly a negative perturbation in C must lead to a positive value of $F_{in}^C - F_{out}^C$).

5 Evolution of a one-variable system perturbed from equilibrium

The evolution of a one-variable system perturbed from equilibrium by a small amount can, in general, be written

$$\dot{C} = kC',\tag{6}$$

If we rewrite this

$$\frac{\partial C'}{\partial t} = kC',$$

it is straightforward to see that the equation can be rearranged:

$$\int \frac{1}{C'} dC' = \int k dt$$

and integrated:

$$\ln(C') + \ln(C_0^{-1}) = kt,$$

where $\ln(C_0^{-1})$ is an arbitrary constant of integration. This can be rearranged:

$$\ln(C_0^{-1}C') = kt,$$

and finally

$$C' = C_0 e^{kt}.\tag{7}$$

We see that if the growth constant k is positive, we have exponential growth, and if k is negative, we have exponential decay. This confirms the result that negative feedbacks are required for an equilibrium to be stable. However, CARE IS NEEDED. Sometimes, even within this course, k (or λ or other symbol) is used as a decay constant rather than a growth constant ($\dot{C} = -kC'$ instead of $\dot{C} = kC'$), in which case positive values would indicate decay and negative values would indicate growth. However, it is always possible to tell how the term is used from the equation and/or physical description of the system.

k (dimension of time^{-1}) gives the rate at which perturbations decay or grow; $1/k$ is the e -folding timescale.

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