Maths 3018/6111 - Numerical Methods

Worksheet 6 - Solutions

Theory

1. Explain the shooting method for BVPs.

The boundary value problem for $y(x)$ with boundary data at $x = a, b$ is converted to an *initial* value problem for $y(x)$ by, at first, guessing the additional (initial) boundary data z at $x = a$ that isrequired for a properly posed (i.e., completely specified) IVP. The IVP can then be solved using any appropriate solver to get some solution $y(x; z)$ that depends on the guessed initial data z. By comparing against the required boundary data at $y(b)$ we can check if we have the correct solution of the original BVP. To be precise, we can write

$$f(z) = y(x; z)|_{x=b} - y(b),$$

a nonlinear equation for z. At the root where $f(z) = 0$ we have the appropriate initial data z such that the solution of the IVP is also a solution of the original BVP. The root of this nonlinear equation can be found using any standard method such as bisection or the secant method.

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2. Give a *complete* algorithm for solving the BVP

$$y'' - 3y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1$$

using the finite difference method. Include the description of the grid, the grid spacing, the treatment of the boundary conditions, the finite difference operators and a description of the linear system to be solved. You do not need to say which method would be used to solve the linear system, but should mention any special properties of the system that might make it easier to solve.

We first choose the grid. We will use $N + 2$ point to cover the domain $x \in [0, 1]$; this implies that we have a grid spacing $h = 1/(N + 1)$ and we can explicitly write the coordinates of the grid points as

$$x_i = hi, \qquad i = 0, 1, \ldots, N + 1.$$

We denote the value of the (approximate finite difference) solution at the grid points as $y_i (\approx y(x_i))$. We will impose the boundary conditions using

$$\begin{aligned} y_0 &= y(0) & \qquad y_{N+1} &= y(1) \\ &= 0 & &= 1. \end{aligned}$$

We will use central differencing which gives

$$\begin{aligned} y'(x)|_{x=x_i} &\approx \frac{y_{i+1} - y_{i-1}}{2h}, \\ y''(x)|_{x=x_i} &\approx \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2}. \end{aligned}$$

We can then substitute all of these definitions into the original definition to find the finite difference equation that holds for the interior points $i = 1, \ldots, N$:

$$y_{i+1}\left(1 - \frac{3}{2}h\right) + y_i\left(-2 + 2h^2\right) + y_{i-1}\left(1 + \frac{3}{2}h\right) = 0.$$

This defines a linear system for the unknowns $y_i, i = 1, \ldots, N$ of the form

$$T\mathbf{y} = \mathbf{f}.$$

We can see that the matrix T is tridiagonal and has the form

$$T = \begin{pmatrix} -2+2h^2 & 1-\frac{3}{2}h & 0 & 0 & 0 & \dots & 0 \\ 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h^2 & 0 & 0 & \dots & 0 \\ 0 & 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \dots & 0 \\ 0 & \dots & 0 & 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h & 0 \\ 0 & \dots & \dots & 0 & 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h \\ 0 & \dots & \dots & \dots & 0 & 1+\frac{3}{2}h & -2+2h^2 \end{pmatrix}$$

The right hand side vector results from the boundary data and is

$$\begin{aligned} \mathbf{f} &= \begin{pmatrix} -\left(1+\frac{3}{2}h\right)y_0 \\ 0 \\ \vdots \\ 0 \\ -\left(1-\frac{3}{2}h\right)y_{N+1} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\left(1-\frac{3}{2}h\right) \end{pmatrix}. \end{aligned}$$

As the system is given by a tridiagonal matrix it is simple and cheap to solve using, e.g., the Thomas algorithm.

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3. Explain how your algorithm would have to be modified to solve the BVP where the boundary condition at $x = 1$ becomes the Neumann condition

$$y'(1) = 1 + \frac{e}{e-1}.$$

First a finite difference representation of the boundary condition is required. A first order representation would be to use backward differencing

$$\frac{y_{N+1} - y_N}{h} = 1 + \frac{e}{e-1}.$$

This can be rearranged to give

$$y_{N+1} = y_N + h\left(1 + \frac{e}{e-1}\right).$$

So now whenever we replaced $y(1)$ as represented by y_{N+1} by the boundary value in the previous algorithm we must instead replace it with the above equation which uses the known boundary data and unknown interior values.

Explicitly, this modifies the matrix T to

$$T = \begin{pmatrix} -2+2h^2 & 1-\frac{3}{2}h & 0 & 0 & 0 & \dots & 0 \\ 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h & 0 & 0 & \dots & 0 \\ 0 & 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \dots & 0 \\ 0 & \dots & 0 & 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h & 0 \\ 0 & \dots & \dots & 0 & 1+\frac{3}{2}h & -2+2h^2 & 1-\frac{3}{2}h \\ 0 & \dots & \dots & \dots & 0 & 1+\frac{3}{2}h & -2+2h^2+\left(1-\frac{3}{2}h\right) \end{pmatrix}$$

and the right hand side vector $\mathbf{f}$ to

$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\left(1-\frac{3}{2}h\right)h\left(1+\frac{e}{e-1}\right) \end{pmatrix}.$$
