Maths 3018/6111 - Numerical Methods

Worksheet 7

Theory

1. Find the linear system that results from using central differencing to solve the elliptic equation

$$u_{xx} + u_{yy} = \sin(\pi y)\left(2 - 6x - (\pi x)^2(1 - x)\right).$$

   Central differencing, trivial boundary conditions, and a 3×2 grid should be used.

2. Explain the strategy for solving evolutionary PDEs using finite differencing methods.

3. Write out a FTCS method for the equations

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \sin(x), & x &\in [0, 1], & y(t, 0) &= 0 = y(t, 1), \\ \partial_t y + \partial_x y &= e^{-(x-1/2)^2}, & x &\in [0, 1], & y(t, 0) &= 0 = y(t, 1). \end{aligned}$$

   In both cases the initial data should be taken to be $y(0, x) = f(x)$. Full details of the grid, the discrete initial and boundary conditions, and the discrete update algorithm should be given.

4. Using von Neumann analysis, find when the BTCS method is stable for the advection equation.

Coding

1. Implement the finite difference method above to solve the elliptic equation

$$u_{xx} + u_{yy} = \sin(\pi y)\left(2 - 6x - (\pi x)^2(1 - x)\right).$$

   Check how it converges with resolution (the exact solution is $u = x^2(1 - x)\sin(\pi y)$.

2. Implement a FTBS algorithm for the advection equation with periodic boundaries. Use a sine wave on $x \in [0, 1]$ as initial data. Check how it converges with resolution.

3. [Additional] Apply the FTBS algorithm to the traffic flow equation

$$\partial_t y + \partial_x (y(1 - y)) = 0.$$

   Use initial data $y(0, x) = \exp(-x^2)$, $y(0, x) = 0.2 + 1.2H(x)$ and $y(0, x) = 1 - H(x)$ on $x \in [-10, 10]$ where $H(x)$ is the Heaviside step function. By comparing against the results for Burger's equation, explain your results. You may want to restrict to $t \leq 0.5$.