Maths 3018/6111 - Numerical Methods

Worksheet 1 - Solutions

Theory

1. Write down the 1, 2 and ∞ vector norms of

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ -3 \\ 1 \end{pmatrix}.$$

We know that the 1-norm is the sum of the absolute values, so

$$\begin{aligned} |\mathbf{v}_1|_1 &= 1 + 3 + 1 = 5, \\ |\mathbf{v}_2|_1 &= 1 + 2 = 3, \\ |\mathbf{v}_3|_1 &= 1 + 6 + 3 + 1 = 11. \end{aligned}$$

The 2-norm is the square root of the sum of the squares, so

$$\begin{aligned} |\mathbf{v}_1|_2 &= \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11} \approx 3.3166, \\ |\mathbf{v}_2|_2 &= \sqrt{1^2 + 2^2} = \sqrt{5} \approx 2.2361, \\ |\mathbf{v}_3|_2 &= \sqrt{1^2 + 6^2 + 3^2 + 1^2} = \sqrt{47} \approx 6.8557. \end{aligned}$$

The infinity norm is the maximum absolute value, so

$$\begin{aligned} |\mathbf{v}_1|_\infty &= 3 \\ |\mathbf{v}_2|_\infty &= 2 \\ |\mathbf{v}_3|_\infty &= 6. \end{aligned}$$

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2. Find the 1 and ∞ matrix norms of

$$A_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & 2 \\ 3 & 6 \end{pmatrix}$$

The 1-norm of a matrix is the maximum of the 1-norms of the column vectors. For A_1 the 1-norms are 4 and 6 respectively. For A_2 they are 6 and 8 respectively. So we have

$$\begin{aligned} \|A_1\|_1 &= 6, \\ \|A_2\|_2 &= 8. \end{aligned}$$

The infinity norm of a matrix is the maximum of the 1-norms of the row vectors. For A_1 the 1-norms are 3 and 7 respectively. For A_2 they are 5 and 9 respectively. So we have

$$\begin{aligned} \|A_1\|_\infty &= 7, \\ \|A_2\|_\infty &= 9. \end{aligned}$$

3. Find the condition numbers of the above matrices. What does this suggest about the numerical behaviour of an algorithm that used such a matrix?

The inverse matrices are

$$A_1^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix},$$
$$A_2^{-1} = -\frac{1}{24}\begin{pmatrix} 6 & -2 \\ -3 & -3 \end{pmatrix}.$$

It follows that the matrix norms of the inverse matrices are

$$\|A_1^{-1}\|_1 = \frac{7}{2}, \qquad \|A_2^{-1}\|_1 = \frac{3}{8},$$
$$\|A_1^{-1}\|_\infty = 3, \qquad \|A_2^{-1}\|_\infty = \frac{1}{3}.$$

Therefore the condition numbers with respect to the 1-norm are

$$\begin{aligned} K(A_1) &= \|A_1\|_1 \|A_1^{-1}\|_1 \\ &= 21, \\ K(A_2) &= \|A_2\|_1 \|A_2^{-1}\|_1 \\ &= 3, \end{aligned}$$

and the condition numbers with respect to the infinity norm are

$$\begin{aligned} K(A_1) &= \|A_1\|_\infty \|A_1^{-1}\|_\infty \\ &= 21, \\ K(A_2) &= \|A_2\|_\infty \|A_2^{-1}\|_\infty \\ &= 3. \end{aligned}$$

In this case they are identical.

This suggests that if the numerical algorithm used the matrix A_1 then the errors intrinsic in the algorithm would increase by a factor of order 10, whilst using A_2 would increase them by a factor order unity. That is, we expect A_2 to be better behaved than A_1 (this is a rather wooly way of putting it).

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4. Explain the difference between direct and indirect methods for solving linear systems. Give an example of when the latter may be more useful.

Standard exam question; see, e.g., 07/08.

Direct methods consist of a finite list of transformations of the original matrix of the coefficients that reduce the linear systems to one that is easily solved. *Indirect or iterative methods*, consist of algorithms that specify a series of steps, possibly infinite, that lead closer and closer to the solution; there may not be a guarantee that they ever exactly reach it. This may not seem a very desirable feature until we remember that we cannot in any case perfectly represent an exact solution: most iterative methods provide us with a highly accurate solution in relatively few iterations.

Large, sparse matrices are ideally solved using iterative methods.
