Maths 3018/6111 - Numerical Methods

Worksheet 5 - Solutions

Theory

1. Explain when multistep methods such as Adams-Bashforth are useful and when multistage methods such as RK methods are better.

Note: There were typos (transposing multistep and multistage) on early versions of this worksheet.

Multistep methods are more computationally efficient (fewer function evaluations) and more accurate than multistage methods. However, they are not self-starting, difficult to adapt to use variable step sizes, and the theory to show that they are stable and convergent is more complex. They are most useful when efficiency is the primary concern and the system is sufficiently well controlled that equally spaced steps can be taken.

In other situations, as discussed on worksheet 4, the self-starting simplicity combined with adaptive stepping means that multistage methods are preferrable.

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2. Compute the coefficients of the AB3 algorithm.

For Adams-Bashforth methods we have

$$y_{n+1} = y_n + b_{k-1} f_n + b_{k-2} f_{n-1} + \cdots + b_0 f_{n+1-k}.$$

Here we have $k = 3$ and so we have

$$y_{n+1} = y_n + h \left[b_2 f_n + b_1 f_{n-1} + b_0 f_{n-2} \right].$$

We want to ensure that this gives an exact approximation of the integral form for polynomials of order $s = 0, \ldots, 2$. That is, we want

$$\int_{x_n}^{x_{n+1}} p_s(x) = h \left[b_2 p_s(x_n) + b_1 p_s(x_{n-1}) + b_0 p_s(x_{n-2}) \right].$$

For simplicity, and without loss of generality, we set $n = 0$, and use the polynomials

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x, \\ p_2(x) &= x(x+h). \end{aligned}$$

We then see that we get

$$
\begin{aligned}
s = 0: && \int_0^h 1 &= h\left[b_2 \times 1 + b_1 \times 1 + b_0 \times 1\right] \\
&& 1 &= b_2 + b_1 + b_0. \\
s = 1: && \int_0^h x &= h\left[b_2 \times 0 + b_1 \times (-h) + b_0 \times (-2h)\right] \\
&& 1/2 &= -b_1 - 2b_0. \\
s = 2: && \int_0^h x(x+h) &= h\left[b_2 \times 0 + b_1 \times 0 + b_0 \times (2h^2)\right] \\
&& 5/6 &= 2b_0.
\end{aligned}
$$

By back substitution we find

$$
b_0 = \frac{5}{12}, \qquad b_1 = -\frac{4}{3}, \qquad b_2 = \frac{23}{12},
$$

which means that the algorithm is

$$
y_{n+1} = y_n + \frac{h}{12}\left[23 f_n - 16 f_{n-1} + 5 f_{n-2}\right].
$$

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3. Explain the meaning of stability, consistency and convergence when applied to numerical methods for IVPs. State the theorem connecting these.

Stability: The numerical solution is bounded at all iterations over a finite interval. I.e., if the true solution is $y(x)$ and $x \in [0, X]$ with X finite, and we use $N+1$ steps with $x_0 = 0$ and $x_N = X$, then $|y_i|$ is finite for all $i = 0, 1, \ldots, N$, irrespective of the value of N.

Consistency: The numerical method is a faithful representation of the differential equation to lowest order in h. That is, if you Taylor expand the numerical difference scheme and let $h \to 0$ you recover the original differential equation independent of the limiting process.

Convergence: If $y(x)$ is the exact solution and $y(x; h)$ the numerical solution using step size h, in the limit as $h \to 0$ the numerical solution is the exact solution:

$$
\lim_{h \to 0} y(x; h) = y(x).
$$

The theorem states that consistency and stability are equivalent to convergence.

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4. Using the stability polynomial and your results above, check the order of accuracy and the stability of the 3 step Adams-Bashforth method.

The coefficients of AB3 in the standard k-step formula notation are

$$\begin{array}{llll} a_3 = 1 & a_2 = -1 & a_1 = 0 & a_0 = 0 \\ b_3 = 0 & b_2 = \dfrac{23}{12} & b_1 = -\dfrac{4}{3} & b_0 = \dfrac{5}{12}. \end{array}$$

Therefore the stability polynomial is

$$p(z) = z^3 - z^2$$

with derivative

$$p'(z) = 3z^2 - 2z$$

and the other required polynomial is

$$q(z) = \frac{1}{12}\left(23z^2 - 16z + 5\right).$$

To check consistency we need that $p(1) = 0$ and $p'(1) = q(1)$, which we check:

$$\begin{aligned} p(1) &= 1 - 1 \\ &= 0. \\ p'(1) - q(1) &= (3-2) - \frac{1}{12}(23 - 16 + 5) \\ &= 1 - \frac{12}{12} \\ &= 0. \end{aligned}$$

So the method is consistent.

To check stability we have to find the roots of the stability polynomial $p(z)$. We write

$$p(z) = z^2(z-1)$$

to see that the roots are 0 (twice) and 1, which means that the method satisfies the *strong* root condition implying both stability and relative stability, meaning it is a useful method.