Large scale structure of metric spaces

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Simple shapes
Large scale structure
How many dimensions?
How many dimensions?
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Definition

Let $X$ be a non-empty set. A \textit{metric} (or a distance function) on $X$ is a map $d : X \times X \to \mathbb{R}$ which satisfied the following properties:

1. \textit{d is positive definite}: for every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. \textit{d is symmetric}: for every $x, y \in X$, $d(x, y) = d(y, x)$.
3. \textit{d satisfies the triangle inequality}: for every $x, y, z \in X$
   $$d(x, z) \leq d(x, y) + d(y, z)$$
Examples of metrics on $\mathbb{R}^n$

The *Euclidean metric* For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ we define

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

The *taxi-cab metric, or the $\ell^1$-metric*:

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n + y_n|$$

The *supremum metric*:

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n + y_n|\}$$
Metric determines shape
Metric determines shape

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Large scale structure
Metric determines shape
Let $X$ be a countable set. A Hilbert space canonically associated with $X$:

$$\ell^2(X) = \left\{ f : X \to \mathbb{C} \mid \sum_{x \in X} |f(x)|^2 < \infty \right\}$$

Canonical orthonormal basis: $\{\delta_x\}$, $f = \sum_{x \in X} f_x \delta_x$, $f_x \in \mathbb{C}$.

Transformations of $X$ give rise to operators on $\ell^2(X)$, e.g., a bijection $\phi : X \to X$ becomes a unitary operator

$$U_\phi : \sum f_x \delta_x \mapsto \sum f_x \delta_{\phi(x)}$$
Graphs provide natural examples of discrete metric spaces:
In a graph, it is natural to define a metric between points to be the length of the *shortest* path between them:
There is no structure theory for discrete metric spaces;
Key features of a space can be determined by studying it from a ‘large distance’
Metrics and function: Network of resistors
A distance between two points can be defined by measuring voltage drop resulting from passing 1 amp of current between them.
The problem of finding the most efficient route between two points depends on the function of the network.

*Picture from physorg.com*
Central core: Normal patients

Lobes: Type I and Type II diabetes, respectively

Conclusion: There are two essentially distinct forms of the disease, one early onset and the other adult onset.

Mathematics for digital economy

Main themes of the proposal

- Geometry
  - Coarse geometry
  - Coarse cohomology
  - Approximate symmetries

- Data sets
  - Synthetic Data
  - Support Vector Machines
  - Clustering and Kernel Methods

- Smarter planet
  - Linked Data
  - Smart grids
  - Sensor Networks

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Large scale structure
The essence of the topological approach is to find the essential core of the system.

Subgraphs consisting of vertices of valency at least: 1,2,3,4.
Basic tools

Definition

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A map \(\phi : X \to Y\) is called distance-preserving if, and only if,
\[d_Y(\phi(x), \phi(y)) = d_X(x, y)\text{ for all }x, y \in X.\]
An isometry is a distance-preserving bijection between two metric spaces.

Example

\(\phi : \mathbb{R}^2 \to \mathbb{C}\) by \((a, b) \mapsto a + bi.\) This is an isometry if \(\mathbb{R}^2\) is equipped with the euclidean metric.
Coarse maps

**Definition**

A map \( f : X \rightarrow Y \) of metric spaces is *coarse* if there exist two functions \( \rho_{\pm} : \mathbb{R} \rightarrow \mathbb{R} \), \( \rho_{\pm}(r) \rightarrow \infty \) as \( r \rightarrow \infty \) such that for all \( x, y \in X \)

\[
\rho_{-}(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_{+}(d_X(x, y))
\]

Coarse maps have a controlled amount of distortion. Maps into spaces of known geometry (e.g., Hilbert spaces) are particularly useful.

The three metrics \( d_{\infty}, d_1, d_2 \) on \( \mathbb{R}^n \) are coarsely equivalent but not isometric.