Problem sheet: Budgets, feedbacks and linear stability

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1 Budgets

A budget is a quantification of the net fluxes of a property into and out of a reservoir, which must be able to explain the net change in that property in the reservoir (e.g. heat in the atmosphere/ocean/ Earth). In general,

$$\frac{\partial}{\partial t}\int C\mathrm{d}V = \Sigma_{i=1}^{n} F_i^C, \tag{1}$$

where C is a conservative property of interest, and we are considering fluxes of C (F^C) into the reservoir. (fluxes out of the reservoir can be represented as negative fluxes into the reservoir). Note that units of F_i^C are the units of C multiplied by the units of volume flux ($m^3\ s^{-1}$) This equation can be easier to use if rewritten:

$$V\dot{C} = F_{in}^C - F_{out}^C, \tag{2}$$

where C is now the mean property value in the reservoir, and we have used:

$$\dot{C} = \frac{\partial C}{\partial t},$$
$$F_{in}^C = \Sigma_{i=1}^{n} F_i^C, \qquad F_i^C > 0$$
$$F_{out}^C = -\Sigma_{i=1}^{n} F_i^C, \qquad F_i^C < 0$$

and assumed that the volume of the reservoir does not change in time. Note that F_{in} and F_{out} can include creation or destruction of the property within the reservoir (e.g. radioactive decay).

1.1 Example: Heat budget of a sphere in a vacuum

If there is incoming shortwave radiation from a distant point source, we can write

$$F_{in}^C = \pi r^2(1-a)I,$$

where I is incoming solar radiation (W m^{-2}), a is albedo and r is the radius of the sphere (m). We assume that there is black-body outgoing radiation from the sphere:

$$F_{out}^C = 4\pi r^2\sigma T^4,$$

where T is temperature (K) and σ is the Stefan-Boltzmann constant (W m^{-2} K^{-4}).

QUESTION (1): Write down the equation for the heat budget of sphere.

ANSWER (1)

$$V\dot{C} = \pi r^2(1 - a)I - 4\pi r^2 \sigma T^4.$$

Here, the property C is heat-content-per-unit-volume, a rather strange property. We'd much rather use temperature T, which is related to heat content by:

$$\mathrm{V}\frac{dC}{d\mathrm{T}} = \mathrm{V}\rho c_p,$$

where ρ is density and c_p is specific heat capacity. Inserting into Eqn (??) yields

$$V\rho c_p \dot{T} = \pi r^2(1 - a)I - 4\pi r^2 \sigma T^4. \quad (3)$$

2 Positive and negative feedbacks

A system has feedbacks if any flux of property C into the reservoir, F_i^c, is a function of C (this includes fluxes out of the system and extends to internal generation terms.) The responce of F_i^c to C need not be immediate; there is feedback even if C only affects F_i^c indirectly. If F_i^c increases as C increases, this is a positive feedback. If F_i^c decreases as C increases, this is a negative feedback. That is:

$$\begin{aligned} \frac{\partial F_i^C}{\partial C} < 0 &\rightarrow \text{Negative feedback} \\ \frac{\partial F_i^C}{\partial C} > 0 &\rightarrow \text{Positive feedback} \end{aligned} \quad (4)$$

Negative F_i^c was reinterpreted as a positive flux out of the system in Equation (2). In this view positive feedback exists when a flux of C into the reservoir increases as C increases, or when a flux of C out of the reservoir decreases as C increases. Similarly, negative feedback exits when a flux of C into the reservoir decreases as C increases, or when a flux of C out of the reservoir increases as C increases.

2.1 Example 1: Outgoing long-wave radiation

QUESTION (2a): Consider Eqn (3). Is the outgoing long-wave radiation term a positive or a negative feedback?

ANSWER (2a)

The last term $-4\pi r^2 \sigma T^4$ indicates a flux of “temperature” (strictly heat) out of the reservoir that increases as temperature T increases. This is a negative feedback, because the increase in temperature causes a negative change in the temperature tendency $\dot{T}$ (i.e. it tends to cause temperature to decrease, in opposition to the original increase).

2.2 Example 2: Temperature–albedo feedback

Consider Eqn (3), again, but now suppose that albedo a is a function of temperature, T and that this function takes the form:

$$\begin{aligned} a &= 0.56, && T \le 271 \\ a = 0.06 + 0.25(273 - T) &= 68.31 - 0.25T, && 271 < T < 273 \\ a &= 0.06, && T \ge 273 \end{aligned}$$

(Such a change in albedo might be caused, for example, by a linear increase change of an ocean-covered sphere from ice-covered to ice-free as temperature increases. Strictly, since ice formation releases heat, and ice melting consumes heat, this would require an extra term in the heat budget, but we will assume that this is negligible.) We can now rewrite Eqn (3):

$$\begin{aligned} V\rho c_p \dot{T} &= 0.44\pi r^2 I - 4\pi r^2 \sigma T^4, && T \le 271 \\ V\rho c_p \dot{T} &= -67.31\pi r^2 I + 0.25\pi r^2 I T - 4\pi r^2 \sigma T^4, && 271 < T < 273 \qquad (5) \\ V\rho c_p \dot{T} &= 0.94\pi r^2 I - 4\pi r^2 \sigma T^4. && T \ge 273 \end{aligned}$$

QUESTION (2b): Has this introduced any further feedbacks to the system? If so, are they positive or negative feedbacks?

ANSWER (2b)

Within the high and low temperature domains, the albedo – and therefore the absorbed solar radiation – is independent of T, so there are no new feedbacks. However the second term in (5) in the intermediate temperature condition, $+0.25\pi r^2 IT$ indicates a flux of temperature into the reservoir that increases as temperature T increases. (This is because albedo decreases as temperature increases, so that a greater proportion of incoming shortwave radiation is absorbed by the reservoir.) This is a positive feedback, because the increase in temperature causes a positive change in the temperature tendency $\dot{T}$ (i.e. it tends to cause temperature to increase further, in addition to the original increase).

To reduce the notation, we now rewrite (5):

$$\begin{aligned} V\rho c_p \dot{T} &= F_{1c} - k_2 T^4, && T \leq 271 \\ V\rho c_p \dot{T} &= F_{1m} + k_1 T - k_2 T^4, && 271 < T < 273 \\ V\rho c_p \dot{T} &= F_{1h} - k_2 T^4. && T \geq 273 \end{aligned} \tag{6}$$

where we have defined

$$F_{1c} = 0.44\pi r^2 I, \quad F_{1m} = -67.31\pi r^2 I, \quad F_{1h} = 0.94\,\pi r^2 I,$$
$$k_1 = 0.25\pi r^2 I, \quad k_2 = 4\pi r^2 \sigma$$

3 System equilibria

"Equilibrium" in system analysis means that the properties of a system are not changing in time (c.f. equilibrium in mechanics, meaning an object's velocity is not changing). This means that any term containing a time derivative must be zero. For the generalised system with a conservative quantity, given in Eqn (2), the criterion for equilibrium is

$$F_{in}^C = F_{out}^C. \tag{7}$$

The fluxes out of the reservoir must equal the fluxes into the reservoir.

3.1 Example: Heat budget of a sphere

QUESTION (3): Rewrite Eqn (5), assuming that the system is at equilibrium.

ANSWER (3)
At equilibrium, $\dot{T}$ is zero, and (5) becomes

$$
\begin{aligned}
F_{1c} - k_2 T_{eq}^4 &= 0, \qquad T_{eq} \leq 271 \\
F_{1m} + k_1 T_{eq} - k_2 T_{eq}^4 &= 0, \qquad 271 < T_{eq} < 273 \\
F_{1h} - k_2 T_{eq}^4 &= 0. \qquad T_{eq} \geq 273
\end{aligned}
\tag{8}
$$

where equilibrium temperatures are denoted by T_{eq} Equilibria in the system may be found be solving Eqn (8) for T_{eq}. At present, the number of equilibria in the low, mid, and high domains of T is unknown.

4 Linear stability of an equilibrium in a one variable system

The most useful test for linear stability of an equilibrium in a system is that a system, if perturbed from equilibrium by a small amount, will return to equilibrium . For a one-variable system, this is equivalent to stating that negative feedbacks must outweigh positive feedbacks at equilibrium, in order for the equilibrium to be stable. This criterion (the Lyapunov criterion) may be expressed:

$$
\begin{aligned}
&\frac{\partial(F_{in}^C - F_{out}^C)}{\partial C} < 0, \rightarrow \text{Stable equilibrium} \\
&\frac{\partial(F_{in}^C - F_{out}^C)}{\partial C} = 0, \rightarrow \text{Critically stable equilibrium} \\
&\frac{\partial(F_{in}^C - F_{out}^C)}{\partial C} > 0, \rightarrow \text{Unstable equilibrium (also metastable equilibrium)}
\end{aligned}
\tag{9}
$$

However, we have already seen that, at equilibrium, $F_{in}^C - F_{out}^C = 0$ (see Eqn 7). Together with Eqn (9), this means that in a stable system, a positive perturbation in C must lead to a negative value of $F_{in}^C - F_{out}^C$ (similarly a negative perturbation in C must lead to a positive value of $F_{in}^C - F_{out}^C$).

4.1 Example: Heat budget of a sphere

Consider a small perturbation from equilibrium. We write

$$T = T_{eq} + T'$$

and

$$T^4 = T_{eq}^4 + 4\,T_{eq}^3 T' + 6\,T_{eq}^2 T'^2 + 4\,T_{eq} T'^3 + T'^4$$

Noting that $T' << T_{eq}$, we ignore terms containing T'^2 or smaller terms:

$$\lim_{T' \to 0} T^4 = T_{eq}^4 + 4\,T_{eq}^3 T',$$

QUESTION (4a): Substitute this approximation into Eqn (5).

ANSWER (4a)

$$
\begin{aligned}
V\rho c_p \dot{T} &= F_{1c} - k_2 T_{eq}^4 - 4k_2 T_{eq}^3 T', && T_{eq} < 271 \\
V\rho c_p \dot{T} &= F_{1m} + k_1 T_{eq} + k_1 T' - k_2 T_{eq}^4 - 4k_2 T_{eq}^3 T', && 271 < T_{eq} < 273 \qquad (10) \\
V\rho c_p \dot{T} &= F_{1h} - k_2 T_{eq}^4 - 4k_2 T_{eq}^3 T'. && T_{eq} > 273
\end{aligned}
$$

QUESTION (4b): Many of the terms appearing in (10) also appear in the equation for the equilibrium of the system (Eq 8). Only the perturbation terms and term containing $\dot{T}$ do not. Therefore simultaneously solving these two equations seems a good idea. Do so

ANSWER (4b)

$$V\rho c_p \dot{T} = -4k_2 T_{eq}^3 T', \qquad T_{eq} < 271 \text{ or } T_{eq} > 273$$
$$V\rho c_p \dot{T} = (k_1 - 4k_2 T_{eq}^3) T', \qquad 271 < T_{eq} < 273 \tag{11}$$

QUESTION (4c): Look at the results you have obtained. Would an equilibrium in the high temperature domain be stable? Would an equilibrium in the intermediate temperature domain be stable? Would an equilibrium in the low temperature domain be stable?

ANSWER (4c)

Because we know that $-4k_2 T_{eq}^3$ must be negative, we know that any equilibrium in the domain $T < 271$ or the domain $T > 273$ must be stable. Mechanistically, we can say that any small increase in temperature will lead to an increase in outgoing radiation, which will return the system to the original equilibrium. Similarly, a small decrease in temperature will lead to a decrease in outgoing radiation, which will return the system to equilibrium. The perturbation will decay.

In the domain $271 < T < 273$, the result is unclear. If $4 k_2 T_{eq}^3 > k_1$, the negative long-wave radiation feedback is stronger than the positive albedo feedback, and the system is stable. However, if $k_1 > 4k_2 T_{eq}^3$, the positive albedo feedback is stronger than the negative long-wave radiation feedback, and the system is unstable. Therefore, a small increase in temperature will lead to a decrease in albedo, which will move the system further away from equilibrium (i.e. warm further). Similarly, a small decrease in temperature will lead to an increase in albedo, which will move the system further away from equilibrium (i.e. cool further). The perturbation will grow.

5 Evolution of a one-variable system perturbed from equilibrium

The evolution of a one-variable system perturbed from equilibrium by a small amount can, in general, be written

$$\dot{C} = \lambda C', \tag{12}$$

(see Eqn 11 for an example). If we rewrite this

$$\frac{\partial C'}{\partial t} = \lambda C',$$

it is straightforward to see that the equation can be rearranged:

$$\int \frac{1}{C'} \mathrm{d}C' = \int \lambda \mathrm{d}t$$

and integrated:

$$\ln (C') + \ln (C_0^{-1}) = \lambda t,$$

where $\ln (C_0^{-1})$ is an arbitrary constant of integration. This can be rearranged:

$$\ln (C_0^{-1} C') = \lambda t,$$

and finally

$$C' = C_0 e^{\lambda t}. \tag{13}$$

We see that if the growth constant λ is positive, we have exponential growth, and if λ is negative, we have exponential decay. This confirms the result that negative feedbacks are required for an equilibrium to be stable. However, CARE IS NEEDED. Sometimes, even within this course, λ (or k or other symbol) is used as a decay constant rather than a growth constant ($\dot{C} = -\lambda C'$ instead of $\dot{C} = \lambda C'$), in which case positive values would indicate decay and negative values would indicate growth. However, it is always possible to tell how it is used from the equation and/or physical description of the system.

λ (dimension of time^{-1}) gives the rate at which perturbations decay or grow; $1/\lambda$ is the *e* - folding timescale.

5.1 Example: Heat budget of a sphere

QUESTION (5a): Write the equation for the evolution of temperature of our sphere, perturbed from equilibrium by a small amount, using Eqn (11).

QUESTION (5b): In the intermediate domain, under what condition would the equilibrium be critically stable, and the evolution equation suggest that the perturbation will neither decay nor grow.

ANSWER (5a)

$$T' = T_0 \exp\left(\frac{-4k_2 T_{eq}^3}{V\rho c_p} t\right), \qquad T_{eq} < 271 \text{ or } T_{eq} > 273$$
$$T' = T_0 \exp\left(\frac{k_1 - 4k_2 T_{eq}^3}{V\rho c_p} t\right), \qquad 271 < T_{eq} < 273 \tag{14}$$

We obtain exponential decay around any equilibria in the domain $T < 271$ or the domain $T > 273$, but exponential growth in the domain $271 < T < 273$ if $k_1 > 4k_2 T_{eq}^3$. The greater the volume of the reservoir, the more slowly T' decays or grows.

ANSWER (5b)

The perturbation neither decays nor grows, and $T' = T_0$, if the growth constant is zero, i.e.:

$$T_{eq}^3 = \frac{k_1}{4k_2}$$

In practice, in this scenario, one should examine the largest term that was neglected in Part 4 (the term containing T'^2) to determine whether a perturbation would decay or grow in this scenario. However, this enters the realm of non-linear stability, which is beyond the scope of this course.

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