QUESTION

Prove the following simple bounds on European call options on an asset that pays no dividends:

(a) $C \leq S$

(b) $C \geq S - k\exp(-r[T-t])$

(c) If two otherwise identical calls have exercise prices K_1 and K_2 with $K_1 < K_2$, then

$$0 \leq C(S,t,K_1) - C(S,t,k_2) \leq k_2 - k_1.$$

(d) If two otherwise identical call options have expiry times T_1 and T_2 and $T_1 < T_2$ then

$$C(S,t,T_1) \leq C(S,t,T_2).$$

ANSWER

(a) $C = \max(S-k, 0)$ so obviously with $k > 0,\ o < C < S$.

(b) Consider $S - c = S - \max(S-k, o)$

Now RHS has spread from $S-(S-k) = k(S > k)$ to $S - 0 = S\ (S < k)$

Therefore $\begin{array}{l} S - c = k, \quad S > k \\ S - c = S, \quad S < k \end{array}$

Therefore $S - c > k$.

(c) $C(S,t,k_1) - C(S,t,k_2) = \max(S-k_1, 0) - \max(S-k_2, 0)$

Now if

$$\begin{aligned} S < k_1 < k_2, RHS &= 0 && (1) \\ k_1 < S < k_2, RHS &= S - k_1 && (2) \\ k_1 < k_2 < S, RHS &= S - k_1 - S + k_2 = k_2 - k_1 && (3) \end{aligned}$$

Now consider (4) versus (5). In (4), $S < k_2$ so (5) is >(4). Also $S - k_1 > 0$ since $S > k_1$ therefore (4)>(3)

Therefore (5)>(4)>(3)

Therefore $) \leq C(S,t,k_1) - C(S,t,k_2) \leq k_2 - k_1$

(d) $T_1 < T_2$

Prove: $C(S, y, T_1) \leq C(S, t, T_2)$

If result is not true (i.e. $C(S, t, T_1) > C(S, t, T_2)$),buy longer-dated call and write the other.

This violates the arbitrage concept since you receive $C(S, t, T_1)$ and payout $C(S, t, T_2)$ making profit of $C(S, t, T_1) - C(S, t, T_2) > 0$ therefore must have $C(S, t, T_1) \leq C(S, t, T_2)$.