QUESTION

Find the most general solution of the Black-Scholes equation that has the special form $V(S,t) = A(t)B(S)$. (Hint: separate the variables).

ANSWER

$V(S,t) = A(t)B(t)$

Substituting this into Black-Scholes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rv = 0$$

$\frac{\partial V}{\partial t} = \frac{dA}{dt}B(S)$
$\frac{\partial V}{\partial S} = A(t)\frac{dB}{dS}$
$\frac{\partial^2 V}{\partial S^2} = A(t)\frac{d^2B}{dS^2}$.

Therefore

$$B\frac{dA}{dt} + \frac{1}{2}\sigma^2 S^2 A\frac{d^2B}{dS^2} + rSA\frac{dB}{dS} - rAB = 0$$

Goal: seperate all the t dependence to one side and all the S dependence to the other side.

To do this divide through by AB and rearrenge:

$$\underbrace{\frac{1}{A}\frac{dA}{dt} - r}_{\text{only } t \text{ dependence (or const.)}} = \underbrace{-\frac{1}{2}\frac{\sigma^2 S^2}{B}\frac{d^2B}{dS^2} - \frac{rS}{B}\frac{dB}{dS}}_{\text{only } S \text{ dependence}}$$

The only way this can happen is if LHS=RHS=const.

Let that constant be C.

Therefore

$$\begin{aligned} \frac{1}{A}\frac{dA}{dt} - r &= C \qquad (1) \\ -\frac{1}{2}\frac{\sigma^2 S^2}{B}\frac{d^2B}{dS^2} - \frac{rS}{B}\frac{dB}{dS} &= C \qquad (2) \end{aligned}$$

Solve(1):

$$\begin{aligned} \frac{1}{A}\frac{dA}{dt} &= r + c \\ \Rightarrow \int \frac{dA}{A} &= \int (r+c)\,dt \\ \Rightarrow \ln A &= (r+c)t + \text{const.} \\ \Rightarrow A &= A(0)e^{(r+c)t} \end{aligned}$$

Solve(2):

$$\frac{\sigma^2}{2}S^2\frac{d^2B}{dS^2}+rS\frac{dB}{dS}+CB=0$$

Euler-equation: try solutions $A=S^n$ with n to be found by substitution.

$$\frac{\sigma^2}{2}S^2n(n-1)S^{n-2}+rSnS^{n-1}+CS^n=0$$

Dividing by S^n

$$\begin{aligned}\frac{\sigma^2}{2}n(n-1)+rn+C &= 0\\ \frac{\sigma^2}{2}n^2+\left(r-\frac{\sigma^2}{2}\right)n+c &= 0\end{aligned}$$

$$n=\frac{\left(\frac{\sigma^2}{2}-r\right)\pm\sqrt{\left(r-\frac{\sigma^2}{2}\right)^2-4\frac{\sigma^2}{2}C}}{\sigma^2}$$

Call these values $n^+,\ n^-$. Then the general solution is

$$B(S)=\alpha r^{n^+}+\beta r^{n^-}$$

$\alpha,\ \beta$ const. Therefore the most general solution to Black-Scholes is

$$\begin{aligned}V=A(t)B(S)=A(0)e^{(r+c)t}\left[\alpha r^{n^+}+\beta r^{n^-}\right]\\ &= e^{(r+c)t}\left(\gamma r^{n^+}+\delta r^{n^-}\right)\end{aligned}$$

$\gamma,\ \delta=$ arbitrary constants to be determined by boundary data. (C is determined in the same way.)