QUESTION

The following table is the Cayley table of a group G of order 8.

	e	g	g^2	g^3	h	hg	hg^2	hg^3
e	e	g	g^2	g^3	h	hg	hg^2	hg^3
g	g	g^2	g^3	e	hg^3	h	hg	hg^2
g^2	g^2	g^3	e	g	hg^2	hg^3	h	hg
g^3	g^3	e	g	g^2	hg	hg^2	hg^3	h
h	h	hg	hg^2	hg^3	g^2	g^3	e	g
hg	hg	hg^2	hg^3	h	g	g^2	g^3	e
hg^2	hg^2	hg^3	h	hg	e	g	g^2	g^3
hg^3	hg^3	h	hg	hg^2	g^3	e	g	g^2

(i) Write each element of the group as a permutation of the set of elements, in disjoint circle notation.

(ii) Give the order of each element.

(iii) Give the sign of each element.

(iv) Find all of the subgroups of G of size 2 and of size 4, giving, for each subgroup H, the set of elements and a generating set.

(v) Show that no two elements of G generate a subgroup isomorphic to Klein's 4-group.

ANSWER

(i)

$$\begin{aligned}
e &= (e)(g)(g^2)\dots(hg^3) \\
g &= (e,g,g^2,g^3)(h,hg^3,hg^2,hg) \\
g^2 &= (e,g^2)(g,g^3)(h,hg^2)(hg,hg^3) \\
g^2 &= (e,g^3,g^2,g)(h,hg,hg^2,hg^3) \\
h &= (e,h,g^2,hg^2)(g,hg,g^3,hg^3) \\
hg &= (e,hg,g^2,hg^3)(g,hg^2,g^3,h) \\
hg^2 &= (e,hg^2,g^2,h)(g,hg^3,g^3,hg) \\
hg^3 &= (e,hg^3,g^2,hg)(g,h,g^3,hg^2)
\end{aligned}$$

(ii) order 1 $\Leftrightarrow$ e

order 2 $\Leftrightarrow$ g^2

order 4 $\Leftrightarrow$ g, g^3, h, hg, hg^2 or hg^3.

(iii) Each element of order 4 is a product of 2 4-cycles (each of which is odd) so it is even g^2 is a product of 4 2-cycles so it too is even. e is always even so $\text{sgn}(x) = 0 \forall x \in G$

(iv) $|H| = 2 \Rightarrow H$ is cyclic of order 2$\Rightarrow H = \langle g \rangle = \{e, g^2\}$

$|H| = 4 \Rightarrow H$ is cyclic of order 4 (since G has only one element of order 2) so $H + \langle g \rangle = \langle g^3 \rangle = \{e, g, g^2, g^3\}$, $H = \langle h \rangle = \langle hg^2 \rangle = \{e, h, g^2, hg^2\}$ or $H = \langle hg \rangle = \langle hg^3 \rangle \{e, hg, g^2, hg^3\}$

(v) Klein's 4-group is an abelian group of order 4 generated by 2 elements of order 2.

G has only one element of order 2 so it has no subgroups isomorphic to Klein's 4-group.