Question

Let m be a parabolic Möbius transformation with fixed point $x \neq \infty$. Show that there exists a unique complex number p so that

$$m(z) = \frac{(1+px)z - px^2}{pz + 1 - px}.$$

Answer

Since we are given a formula, we can check it directly: suppose there are two such ps, call them p_1, p_2; and write:

$$m(z) = \frac{(1+p_1x)z - p_1x^2}{p_1z + 1 - p_1x} = \frac{(1+p_2x)z - p_2x^2}{p_2z + 1 - p_2x}$$

Then,

$$mm^{-1}(z) = \frac{(1 + p_1x - p_2x)z + p_2x^2 - p_1x^2}{(p_1 - p_2)z + (1 - p_1x + p_2x)}$$

Since $mm^{-1}(\infty) = \infty$, the coefficient of z in the denominator is 0, and so $p_1 = p_2$.

As for existence: for all $p \in \mathbf{C}$, all $x \in \mathbf{C}$.

$$m(z) = \frac{(1+px)z - px^2}{pz + 1 - px}$$

is parabolic fixing x (except $p = 0$ when $m(z) = z$).

(Could also explicitly derive the formula for $m(z)$ directly from the information given.)