Applications of Partial Differentiation
Extremes

Question
Find and classify the critical points of the function

$$f(x, y) = xe^{-x^3+y^3}$$

Answer

$$\begin{aligned}
f_1(x, y) &= (1 - 3x^3)e^{-x^3+y^3} \\
f_2(x, y) &= 3xy^2e^{-x^3+y^3} \\
A = f_{11}(x, y) &= 3x^2(3x^3 - 4)e^{-x^3+y^3} \\
B = f_{12}(x, y) &= -3y^2(3x^3 - 1)e^{-x^3+y^3} \\
C = f_{22}(x, y) &= 3xy(3y^3 + 2)e^{-x^3+y^3}
\end{aligned}$$

For critical points: $3x^3 = 1$ and $3xy^2 = 0$. The only critical point is $(3^{-1/3}, 0)$. At that point we have $B = C = 0$ so the second derivative test is inconclusive. However, note that $f(x, y) = f(x, 0)e^{y^3}$, and e^{y^3} has an inflection point at $y = 0$. Therefore $f(x, y)$ has neither a maximum nor a minimum value at $(3^{-1/3}, 0)$, so has a saddle point there.