Applications of Partial Differentiation
Extremes

Question

(a)

$$f(x, y) = (y - x^2)(y - 3x^2)$$

Show that the origin is a critical point of f and that the restriction of f to every straight line through the origin has a local minimum value at the origin.

(i.e. show that $f(x, kx)$ has a local minimum value at $x = 0\ \forall k$, and that $f(0, y)$ has a local minimum value at $y = 0$.)

(b) Is there a local minimum value of $f(x, y)$ at the origin?

(c) On the curve $y = 2x^2$, what happens to f?

What does the second derivative test say about this situation?

Answer

(a)

$$\begin{aligned} f(x, y) &= (y - x^2)(y - 3x^2) = y^2 - 4x^2y + 3x^4 \\ f_1(x, y) &= -8xy + 12x^3 = 4x(3x^2 - 2y) \\ f_2(x, y) &= 2y - 4x^2 \end{aligned}$$

Since $f_1(0, 0) = f_2(0, 0) = 0$, therefore $(0, 0)$ is a critical point of f.

(b) Let $g(x) = f(x, kx) = k^2x^2 - 4kx^3 + 3x^4$. Then

$$\begin{aligned} g'(x) &= 2k^2x - 12kx^2 + 12x^3 \\ g''(x) &= 2k^2 - 24kx + 36x^2. \end{aligned}$$

Since $g'(0) = 0$ and $g''(0) = 2k^2 > 0$ for $k \neq 0$, g has a local minimum value at $x = 0$. Thus $f(x, kx)$ has a local minimum at $x = 0$ if $k \neq 0$.

Since $f(x, 0) = 3x^4$ and $f(0, y) = y^2$ both have local minimum values at $(0, 0)$, f has a local minimum at $(0, 0)$ when restricted to any straight line through the origin.

(c)

However, on the curve $y = 2x^2$ we have

$$f(x, 2x^2) = x^2(-x^2) = -x^4$$

which has a local maximum value at the origin. Therefore f does *not have an (unrestricted) local minimum value at the origin.*

Note that

$$\begin{aligned} A &= f_{11}(0,0) = (-8y + 36x^2)\Big|_{(0,0)} = 0 \\ B &= f_{12}(0,0) = -8x|_{(0,0)} - 0 \end{aligned}$$

Thus $AC = B^2$ *and the second derivative test at the origin is indeterminate.*