

Applications of Partial Differentiation

Extremes

Question

The cost of making a wall for a rectangular box depends on the walls' position. To make the bottom of the box is twice as expensive per unit area as it is to make to top or side walls. Find the dimensions of a box of volume V that minimizes the cost of making the box.

Answer

Let the length, width and height of the box be x , y and z respectively. Then $V = xyz$.

If the top and side walls cost $\mathcal{L}k$ per unit areas, then the total cost of the materials needed for the box is

$$\begin{aligned} C &= 2kxy + kxy + 2kxz + 2kyz \\ &= k \left[3xy + 2(x+y)\frac{V}{xy} \right] \\ &= k \left[3xy + \frac{2V}{x} + \frac{2V}{y} \right] \end{aligned}$$

where $x > 0$ and $y > 0$.

Since $C \rightarrow \infty$ as $x \rightarrow 0+$ or $y \rightarrow 0+$ or $x^2 + y^2 \rightarrow \infty$, C must have a minimum value at a critical point in the first quadrant.

For CP:

$$\begin{aligned} 0 = \frac{\partial C}{\partial x} &= k \left(3y - \frac{2V}{x^2} \right) \\ 0 = \frac{\partial C}{\partial y} &= k \left(3x - \frac{2V}{y^2} \right) \end{aligned}$$

Thus $3x^2y = 2V = 3xy^2$ so that $x = y = (2V/3)^{1/3}$
and $z = V/(2V/3)^{2/3} = (9V/4)^{1/3}$.