Applications of Partial Differentiation
Extremes

Question

Find and classify the critical points of the function

$$f(x, y) = \frac{xy}{2 + x^4 + y^4}$$

Answer

$$\begin{aligned} f_1 &= \frac{(2 + x^4 + y^4)y - xy4x^3}{(2 + x^4 + y^4)^2} \\ &= \frac{y(2 + y^4 - 3x^4)}{(2 + x^4 + y^4)^2} \\ f_2 &= \frac{x(2 + x^4 - 3y^4)}{(2 + x^4 + y^4)^2}. \end{aligned}$$

For critical points, $y(2 + y^4 - 3x^4) = 0$ and $x(2 + x^4 - 3y^4) = 0$.
One critical point is $(0, 0)$. Since $f(0, 0) = 0$ but $f(x, y) > 0$ in the first quadrant and $f(x, y) < 0$ in the second quadrant, $(0, 0)$ must be a saddle point of f.
Any other critical points must satisfy $2 + y^4 - 3x^4 = 0$ and $2 + x^4 - 3y^4 = 0$, that is $y^4 = x^4$, or $y = \pm x$. Thus $2 - 2x^4 = 0$ and $x = \pm 1$.
Therefore there are four other critical points: $(1, 1)$, $(-1, -1)$, $(1, -1)$ and $(-1, 1)$.
f is positive at the first two of these, and negative at the other two.
Since $f(x, y) \to 0$ as $x^2 + y^2 \to \infty$, f must have maximum values at $(1, 1)$ and $(-1, -1)$, and minimum values at $(1, -1)$ and $(-1, 1)$¿