Question

Find **all** the solutions ($y(x)$ for $x \geq 0$) of the following differential equations with the given initial condition. (You may wish to sketch the direction field and the solution curves in each case to help indicate if you have missed any solutions. Be particularly aware of possible division by zero in your algebra).

1. $\left(\dfrac{dy}{dx}\right)^2 = y \qquad y(0) = 1 \qquad (*)$

2. $\left(\dfrac{dy}{dx}\right)^2 = y \qquad y(0) = 0 \qquad (*)$

3. $\left(\dfrac{dy}{dx}\right)^4 = x^2(y-1) \qquad y(0) = 1$

Answer

a) Can be written as either i) $\dfrac{dy}{dx} = +y^{\frac{1}{2}}$ or ii) $\dfrac{dy}{dx} = -y^{\frac{1}{2}}$ $(y \geq 0)$
which are separable, so if $y \neq 0$

i) $\displaystyle\int y^{-\frac{1}{2}} dy = \int dx \Rightarrow 2y^{\frac{1}{2}} = x + A$ use $y(0) = 1 \Rightarrow A = 2$

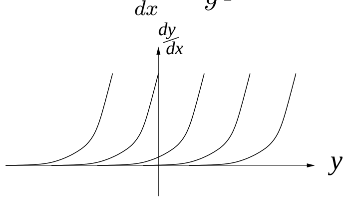
$\Rightarrow 2y^{\frac{1}{2}} = x + 2 \Rightarrow y = \left(\dfrac{x}{2} + 1\right)^2$

ii) $\displaystyle\int y^{-\frac{1}{2}} dy = -\int dx \Rightarrow 2y^{\frac{1}{2}} = -x + A$ use $y(0) = 1 \Rightarrow A = 2$

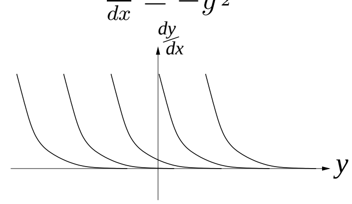
$\Rightarrow 2y^{\frac{1}{2}} = -x + 2 \Rightarrow y = \left(\dfrac{x}{2} - 1\right)^2$

note: $y = 0$ when $x = 2$

$\frac{dy}{dx} = y^{\frac{1}{2}}$



$\frac{dy}{dx} = -y^{\frac{1}{2}}$



Hence two possible solutions are:

1) $y = \left(\frac{x}{2} + 1\right)^2 \qquad x \geq 0$

2) $y = \begin{cases} \left(\frac{x}{2} - 1\right)^2 & 0 \leq x \leq 2 \\ 0 & x > 2 \end{cases}$

b) Use the direction field above and consider $y(0) = 0$

If $\frac{dy}{dx} = -y^{\frac{1}{2}} \Rightarrow y \equiv 0 \quad x \geq 0$ however if $\frac{dy}{dx} = y^{\frac{1}{2}}$
there are a whole set of solutions given by

$$y = \begin{cases} 0 & 0 \leq x \leq A \\ \left(\frac{x}{2} - \frac{A}{2}\right)^2 & x > A \end{cases}$$

and A can be any positive constant.

c) Consider the positive roots first.

$\frac{dy}{dx} = x^{\frac{1}{2}}(y-1)^{\frac{1}{4}} \quad y \geq 1$

$\int (y-1)^{\frac{-1}{4}} dy = \int x^{\frac{1}{2}} dx \quad$ for $y \neq 1$

$\frac{4}{3}(y-1)^{\frac{3}{4}} = \frac{2}{3}x^{\frac{3}{2}} + A$ note $y = 1$ is a solution,

so all the solutions are given by:

$y(x) = 1$ for $0 \leq x \leq B$

$\frac{4}{3}(y-1)^{\frac{3}{4}} = \frac{2}{3}x^{\frac{3}{2}} + A$ for $x \geq B$ with $y(B) = 1$

$\frac{4}{3}(y-1)^{\frac{3}{4}} = \frac{2}{3}(x^{\frac{3}{2}} - B^{\frac{3}{2}})$

$$\Rightarrow y(x) = \begin{cases} 1 & 0 \leq x \leq B \\ 1 + (\frac{1}{2}(x^{\frac{3}{2}} - B^{\frac{3}{2}}))^{\frac{4}{3}} & x \geq B \end{cases}$$

for any constant $B \geq 0$

Now consider the negative roots

$\frac{dy}{dx} = -x^{\frac{1}{2}}(y-1)^{\frac{1}{4}} \quad y \geq 1 \quad x \geq 0$ the only solution with $y(0) = 1$ is
$y \equiv 1$ for all $x \geq 0$ as in 4b).