Question
By evaluating the relevant determinant, investigate for various values of a and b the type of solution set of the equations

$$\begin{aligned} x + 3y - 2z &= 7 \\ ax + 6y - 4z &= 2 - 3b \\ 2x + 6y + bz &= 14 \end{aligned}$$

Answer
The equations can be written as:

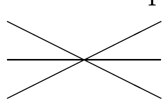
$$\begin{pmatrix} 1 & 3 & -2 \\ a & 6 & -4 \\ 2 & 6 & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 2-3b \\ 14 \end{pmatrix}$$

whether we solve by inverting the coefficient matrix or with a determinant method (Cramer's rule), we have to work out its determinant.
If this determinant=0 we don't have an inverse, or alternatively,
Cramer's rule will fail.
Thus we need to work out

$$\begin{aligned} \begin{vmatrix} 1 & 3 & -2 \\ a & 6 & -4 \\ 2 & 6 & b \end{vmatrix} &= 1 \times \begin{vmatrix} 6 & -4 \\ 6 & b \end{vmatrix} - 3 \begin{vmatrix} a & -4 \\ 2 & b \end{vmatrix} + (-2) \begin{vmatrix} a & 6 \\ 2 & 6 \end{vmatrix} \\ &= 6b + 24 - 3ab - 24 - 12a + 24 \\ &= 24 - 12a + 6b - 3ab \\ &= 3(8 - 4a + 2b - ab) \\ &= 3(2-a)(4+b) \end{aligned}$$

So

(i) If $a \neq 2$ and $b \neq -4$ there is a unique solution, since $det \neq 0$. The planes meet at a point



(ii) If $a = 2$ and $b \neq -4$ then $det = 0$. The system is

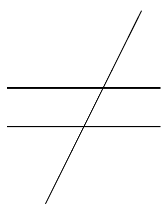
$$\begin{aligned} x + 3y - 2z &= 7 \quad (1) \\ 2x + 6y - 4z &= 2 - 3b \quad (2) \\ 2x + 6y + bz &= 14 \quad (3) \end{aligned}$$

$(2) \Rightarrow x + 3y - 2z = 1 - 3\frac{b}{2}$

cf $(1) \Rightarrow x + 3y - 2z = 7$

These are two parallel planes if $1 - 3\frac{b}{2} \neq 7 \Rightarrow b \neq 4$

So they're parallel and non intersecting:



parallel planes

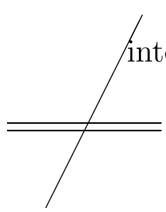
(iii) If $b = -4$ but $a \neq 2$ we have

$$\begin{aligned} x + 3y - 2z &= 7 \\ ax + 6y - 4z &= 14 \\ 2x + 6y - 4z &= 14 \end{aligned}$$

(3) is twice (1)

(2) and (3) are not the same since $a \neq 2$

We have two coincident planes with an intersecting plane.



intersecting plane

2 coincident planes

Thus the solution is a line of points not a single one.

(iv) If $a = 2$ and $b = -4$ we get

$$\left.\begin{aligned} x + 3y - 2z &= 7 \\ ax + 6y - 4z &= 14 \\ 2x + 6y - 4z &= 14 \end{aligned}\right\} \text{all the same equation!!!}$$

We have 3-coincident planes.

$\equiv$ 3 coincident planes

So a plane of x, y, z solutions, not a single point.