Question

Prove that there does not exist an order on the complex numbers **C** so that **C** becomes an ordered field.

Answer

Suppose there were such an order on **C**, and denote it by $<$. Compare 0 and i. Since $0 \neq \mathrm{i}$, it must be that either $0 < \mathrm{i}$ or $\mathrm{i} < 0$.

Suppose that $0 < \mathrm{i}$. Multiplying both sides by i and remembering that $0 < \mathrm{i}$, we see that $0 \cdot \mathrm{i} < \mathrm{i} \cdot \mathrm{i}$, which simplifies to $0 < -1$. Adding 1 to both sides, we see that $1 < 0$. Again multiplying both sides by i and remembering that $0 < \mathrm{i}$, we see that $1 \cdot \mathrm{i} < 0 \cdot \mathrm{i}$, which simplifies to $\mathrm{i} < 0$. Hence, if $0 < \mathrm{i}$, then $\mathrm{i} < 0$, contradicting the second condition in the definition of an order.

Suppose now that $\mathrm{i} < 0$. Adding the additive inverse $-\mathrm{i}$ of i to both sides, we get that $0 < -\mathrm{i}$. Multiplying both sides by $-\mathrm{i}$, we get that $0 \cdot (-\mathrm{i}) < (-\mathrm{i}) \cdot (-\mathrm{i})$, and so $0 < -1$. Multiplying both sides by $-\mathrm{i}$ again, we get that $0 < (-1) \cdot (-\mathrm{i}) = \mathrm{i}$. Hence, if $\mathrm{i} < 0$, then then $0 > \mathrm{i}$, again contradicting the second condition in the definition of an order.

Hence, since we have that neither $0 < \mathrm{i}$ nor $\mathrm{i} < 0$, we see that there cannot exist an order on **C** that makes **C** into an ordered field.