Question

(a) Suppose that $U(S,t)$ satisfies the Black-Scholes equation. Show that if V is defined by

$$U(S,t) = S^n V(\eta, t)$$

where $\eta = K/S$ and K and n are constant then

$$V_t + \frac{1}{2}\sigma^2 V_{\eta\eta} + r\eta V_\eta - rV = 0$$

provided n takes a particular value (which you should determine).

(b) A European DOWN-AND-OUT Call with strike E, expiry T and barrier X is identical to a European Call option *except* for the fact that the option cannot be exercised if the price of the underlying ever drops below X. Explain briefly why the value $D(S,t)$ of such an option must satisfy $D(X,t) = 0$ and $D(X,T) = \max[S - E, 0]$. Using the result of part (a) or otherwise show that, if $C(S,t)$ denotes the value of a European Call option with strike E and expiry T, then

$$D(S,t) = C(S,t) - AS^{1-2r/\sigma^2} C(K/S, t)$$

where A and K are constants (which you should determine).

(c) By considering the payoff of a portfolio which is long one down-and-out Call and long one down-and-in Call, determine the value of a down-and-in Call.

Answer

(a) Since U satisfies Black-Scholes we have

$U_t + \frac{1}{2}\sigma^2 S^2 U_{SS} + rSU_S - rU = 0.$

Now put $U = S^n V(\eta, t)$ where $\eta = K/S$.

Then

$$\begin{aligned}
U_t &= (S^n V)_\eta \eta_t + (S^n V)_t t_t = 0 + S^n V_t \\
&= S^n V_t \\
U_S &= nS^{n-1}V + S^n V_S = nS^{n-1}V + S^n V_\eta \eta_S \\
&= nS^{n-1}V - \eta S^{n-1} V_\eta
\end{aligned}$$

also

$$\begin{aligned}
U_{SS} &= n(n-1)S^{n-2}V + nS^{n-1}V_\eta \left(-\frac{K}{S^2}\right) \\
&\quad -(n-2)KS^{n-3}V_\eta - KS^{n-2}V_{\eta\eta}\left(-\frac{K}{S^2}\right) \\
&= n(n-1)S^{n-2}V - \eta S^{n-2} nB_\eta - (n-2)S^{n-2}\eta V_\eta \\
&\quad + S^{n-2}\eta^2 V_{\eta\eta}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
S^n V_t &+ \frac{1}{2}\sigma^2 S^2 [n(n-1)S^{n-2}V - S^{n-2} n\eta V_\eta - (n-2)S^{n-2}\eta V \\
&+ \eta + S^{n-2}\eta^2 V_{\eta\eta}] \\
&+ rS[nS^{n-1}V - S^{n-1}\eta V_\eta] - rS^n V = 0
\end{aligned}$$

Canceling S^n and re-arranging gives

$$V_t + V_{\eta\eta}\left[\eta^2 \frac{1}{2}\sigma^2\right] + \left[-\frac{n\sigma^2}{2} - \frac{\sigma^2}{2}(n-2) - r\right]\eta V_\eta$$

$$+V\left[\frac{1}{2}\sigma^2(n-1)N + rn - r\right] = 0$$

So to get back to Black-Scholes again we need n such that

$$\begin{aligned}
-\frac{n\sigma^2}{2} - \frac{\sigma^2}{2}n + \sigma^2 &= 2r \\
\Rightarrow n &= 1 - \frac{2r}{\sigma^2}.
\end{aligned}$$

With this value, the coefficient of V becomes

$$\frac{\sigma^2}{2}\left(1-\frac{2r}{\sigma^2}\right)\left(-\frac{2r}{\sigma^2}\right)-\frac{2r^2}{\sigma^2}=-r$$

Thus with $n = 1 - 2r/\sigma^2$ we DO get back to B/Scholes.

(b) For a European Down-and-out call the payoff at expiry is the same as a vanilla call IF the option is exercised. Thus at expiry T

$$D(S,T) = \max(\mathrm{S-E},0)$$

Also, the option becomes worthless $\forall t$ the instant that the share price hits $S = X$ and thus

$$D(X,t) = 0.$$

Now consider $D(S,t) = C(S,t) - AS^{1-\frac{2r}{\sigma^2}}C\left(\frac{K}{S},t\right)$.

We have to show that this satisfies 3 things:-

(i) Must satisfy Black-Scholes. Well $C(S,t)$ does by definition and by part (a) of this question so does $S^{1-\frac{2r}{\sigma^2}}C(K/S,t)$.
The linearity of Black-Scholes now ensures that we may add solutions $\Rightarrow$ Black-Scholes is satisfied.

(ii) We must ensure that $D(X,t) = 0\ \forall t$. Now

$$D(X,t) = C(X,t) - AX^{1-\frac{2r}{\sigma^2}}C(K/X,t)$$

and clearly we can fix this up to be zero if we choose

$$A = X^{-\left(1-\frac{2r}{\sigma^2}\right)}, \quad K = X^2$$

(iii) We must ensure finally that $B(S,T) = \max(\mathrm{S-E},0)$.
But

$$\begin{aligned} B(S,T) &= C(S,T) - (S/X)^{1-\frac{2r}{\sigma^2}}C(X^2/S,T) \\ &= \max(\mathrm{S-E},0) - (\mathrm{S/X})^{1-\frac{2r}{\sigma^2}}\max(\mathrm{X^2/S-E},0) \end{aligned}$$

But for sure $S > X$ and $E > X$ so $X^2/S - E <)$

$$\Rightarrow B(S,T) = \max(\mathrm{S-E},0)$$

as it should.

(c) Let $\Pi - D_i(S,t) + D_0(S,t)$

Then obviously

$$\Pi = C(S,t)$$

since whether the barrier is triggered or not the option will be the same as a European call.

$$\begin{aligned}
\Rightarrow D_i(S,t) + D_0(S,t) &= C(S,t) \\
\Rightarrow D_i(S,t) &= C(S,t) \\
&\quad - \left[C(S,t) - (S/X)^{1-\frac{2r}{\sigma^2}} C(X^2/S,t)\right] \\
D_i(S,t) &= \frac{S}{X}^{1-\frac{2r}{\sigma^2}} C(X^2/S,t)
\end{aligned}$$