Question

In this question YOU MAY ASSUME

(i) that small changes df in the function $f(S,t)$ are related to small changes in S and t by Taylor's theorem so that

$$df = f_S dS + f_t dt + \frac{1}{2} f_{SS} dS^2 + f_{St} dS dt + \frac{1}{2} f_{tt} dt^2 + \cdots$$

(ii) that S follows the lognormal random walk

$$\frac{dS}{S} = r dt + \sigma dX$$

where r and σ are constants and X is a random variable,

(iii) that $dX^2 \to dt$ as $dt \to 0$.

(a) Derive Itô's lemma in the form

$$df = \sigma S f_S dX + \left(f_t + rS f_S + \frac{1}{2}\sigma^2 S^2 f_{SS} \right) dt$$

and comment briefly on whether or not your derivation is rigorous.

(b) Denote the fair value of an option by $V(S,t)$. By constructing a portfolio $\Pi = V - \Delta S$ where Δ is to be determined, show that V satisfies the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

(c) A PERPETUAL option is one whose value does not depends upon time. Find the most general solution for the value of a perpetual option and show that the value of a perpetual Put is given by

$$V = AS^{-2r/\sigma^2}$$

where A is a constant that depends on the specific details of the option.

Answer

(a) We have, by Taylor's theorem:-

$$df = f_S dS + f_t dt + \frac{1}{2} f_{SS} dS^2 + f_{St} dS dt + \frac{1}{2} f_{tt} dt^2 + \cdots$$

and

$$\begin{aligned} dS &= S\mu dt + S\sigma dX \\ \Rightarrow dS^2 &= S^2\mu^2 dt^2 + 2S^2\mu\sigma dt dX + S^2\sigma^2 dX^2 \end{aligned}$$

But now as $dt \to 0$, $dX^2 \to dt$

$$\begin{aligned} \Rightarrow dS^2 &= S^2\sigma^2 dt + 2\sigma\mu S^2 (dt)^{3/2} + S^2\mu^2 dt^2 \\ &= S^2\sigma^2 dt + O(dt^{3/2}) \\ \Rightarrow df &= f_S[S\mu dt + S\sigma dX] + f_t dt + \frac{1}{2}(f_{SS}\sigma^2 S^2 dt \\ & \quad + O(dt^{3/2})) + O(dt^{3/2}) \end{aligned}$$

and so, to leading order,

$$\begin{aligned} df &= f_S[S\mu dt + S\sigma dX] + f_t dt + \frac{1}{2}\sigma^2 S^2 f_{SS} dt \\ &= S\sigma F_S dX + (F_t + \mu S f_S + \frac{1}{2}\sigma^2 S^2 f_{SS}) dt \quad - \quad \text{ITO's lemma} \end{aligned}$$

The derivation is not very rigorous at all - it started from Taylor's theorem which is valid for smooth functions - and S follows a random walk!

(b) Now consider the portfolio $\Pi = V - S\Delta$.

We have

$$\begin{aligned} d\Pi &= dV - \Delta dS = S\sigma V_S dX + \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS}\right) dt \\ & \quad -\Delta(\mu D st + \sigma S dX) \\ d\Pi &= (S\sigma V_S - \Delta S\sigma) dX \\ & \quad + \left(V_t + \mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{SS} - \Delta S\mu\right) dt \end{aligned}$$

All the randomness in Π may thus be eliminated by choosing $\Delta = V_S$, in which case we find that

$$\begin{aligned} d\Pi &= \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - S\mu V_S\right) dt \\ &= \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right) st. \end{aligned}$$

We now appeal to arbitrage: presumably the option must be neither more nor less valuable than a risk free investment, otherwise one or the other would never be used. So the above must be equal to the return in time dt of an amount Π invested in a risk free portfolio. Thus

$$\begin{aligned} r\Pi dt &= \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right) dt \\ r(V - S\Delta) &= V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} \\ r(V - SV_S) &= V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} \quad \Rightarrow \ \text{Black} - \text{Scholes} \end{aligned}$$

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

(c) For a perpetual option we have $V_t = 0 \Rightarrow \ V = V(S)$ only.

$\Rightarrow$ Black-Scholes becomes

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

This is Euler's equation, so for solution try $V = S^k$

$$\begin{aligned} \frac{1}{2}\sigma^2 S^2 k(k-1)S^{k-2} + rSkS^{k-1} - rS^k &= 0 \\ S^k\left(\frac{1}{2}\sigma^2 k(k-1) + rk - r\right) &= 0 \end{aligned}$$

So for a solution we need $\frac{1}{2}\sigma^2 k(k-1) + (k-1)r = 0.$

$$\begin{aligned} \Rightarrow \quad \text{either} \quad & k = 1 \\ \text{or} \quad & \tfrac{1}{2}\sigma^2 k + r = 0 \\ \Rightarrow \quad & k = -2r/\sigma^2 \end{aligned}$$

Thus the most general solution for a perpetual option is

$$V = AS + BS^{-2r/\sigma^2}$$

where A and B are arbitrary constants.

Now consider an American (or European) Put which is perpetual! Clearly as $S \to \infty$ the option becomes more and more worthless, since the chance of exercising it becomes less and less. $\Rightarrow A = 0$

$\Rightarrow$ for some $\overline{A}$

$$V + \overline{A}S^{-2r/\sigma^2}$$