Question

Consider a portfolio Π which is composed of a proportion $\lambda < 1$ of a risk free asset S_0 with associated return R_0 and a proportion $1-\lambda$ of a risky portfolio S_1 with associated return R_1 and variance σ_1^2. Show that as λ varies, Π lies along a straight line in the risk/reward diagram, the line having slope $\theta = (R_1 - R_0)/\sigma_1$. Explain briefly why this implies that the problem of finding the capital market line reduces to that of maximizing θ over all risky portfolios.

Now consider a scenario where there are three risky assets S_1, S_2 and S_3 with respective expected returns

$$R_1 = 0.08, \quad R_2 = 0.10, \quad R_3 = 0.12.$$

The variances and covariances between the assets are given by

$$\begin{aligned} \sigma_1^2 &= 0.008 \\ \sigma_{12} &= 0.004 \\ \sigma_{13} &= 0 \\ \sigma_2^2 &= 0.006 \\ \sigma_{23} &= 0.002 \\ \sigma_3^2 &= 0.008 \end{aligned}$$

and the risk free rate is 0.05. Short selling and borrowing are allowed. Show that the optimal portfolio of risky assets consists of investing proportions 1/11, 4/11 and 6/11 of one's total wealth in S_1, S_2 and S_3 respectively. Show that the associated risk and return are $\sqrt{520/121} \sim 2.07\%$ and $120/11 \sim 10.91\%$ respectively, and the market price of risk is

$$\theta = \frac{\sqrt{130}}{4} \sim 2.85$$

Answer
We have $\Pi = \lambda S_0 + (1-\lambda)S_1$ and thus

$$R_\Pi = \lambda R_0 + (1-\lambda)R_1.$$

For the portfolio variance σ_Π^2 we have

$$\sigma_\Pi^2 = \lambda^2\sigma_0^2 + 2\lambda(1-\lambda)\sigma_{01} + (1-\lambda)^2\sigma_1^2.$$

But S_0 is riskless, so by definition $\sigma_0 = 0$, $\sigma_{01} = 1$.
Thus

$$\begin{aligned} \sigma_\Pi^2 &= (1-\lambda)^2\sigma_1^2 \\ \Rightarrow \sigma_\Pi &= (1-\lambda)\sigma_1 \end{aligned}$$

Now

$$\begin{aligned} \lambda < 1 &\Rightarrow 1-\lambda > 0 \\ &\Rightarrow \sigma_\Pi = (1-\lambda)\sigma_1 \\ &\Rightarrow \lambda = 1 - \frac{\sigma_\Pi}{\sigma_1} \end{aligned}$$

Thus

$$R_\Pi = \left(1 - \frac{\sigma_\Pi}{\sigma_1}\right) R_0 + \frac{\sigma_\Pi}{\sigma_1} R_1 = R_0 + \sigma_\Pi \left(\frac{R_1 - R_0}{\sigma_1}\right)$$

A straight line of slope $(R_1 - R_0)/\sigma_1$ as required.
Now the CML is just the straight line through the risk free return R_0 at σ_Π which is tangent to the boundary of the (convex) opportunity slope; obviously therefore it is the line that passes through $(R_0, 0)$ and some risky portfolio that has positive slope.
Now we have to maximize $\theta = (R_\Pi - R_0)/\sigma_\Pi$ over all possible risky portfolios where

$$\begin{aligned} \Pi &= X_1S_1 + X_2S_2 + X_3S_3 \\ with & \quad X_1 + X_2 + X_3 - 1. \end{aligned}$$

Now

$$R_\Pi = X_1R_1 + X_2R_2 + X_3R_3 = \frac{1}{100}(8X_1 + 10X_2 + 12X_3)$$

and

$$\begin{aligned} \sigma_\Pi = & (X_1^2\sigma_1^2 + 2X_1X_2\sigma_{12} + X_2^2\sigma_2^2 + 2X_1X_3\sigma_{13} + X_3^2\sigma_3^2 \\ & +2X_2X_3\sigma_{23})^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= (X_1^2 + 8X_1X_2 + 6X_2^2 + 8X_3^2 \\
&\quad +4X_2X_3)^{\frac{1}{2}} \left(\frac{10^{\frac{1}{2}}}{100}\right) \\
&= 10^{\frac{1}{2}} \frac{\alpha^{\frac{1}{2}}}{100} \text{ say.}
\end{aligned}
$$

Now we have to maximize $\theta = \dfrac{(8-5)X_1 + (10-5)X_2 + (12-5)X_3}{\alpha^{1/2}}$.

$$
\text{i.e.} \quad \theta = \frac{3\mathrm{X}_1 + 5\mathrm{X}_2 + 7\mathrm{X}_3}{\alpha^{1/2}}.
$$

Let $3X - 1 + 5X_2 + 7X_3 = \beta$. Then

$$
\begin{aligned}
\frac{\partial\theta}{\partial X_1} &= \frac{3\alpha^{1/2} - \beta\frac{1}{2}\alpha^{-1/2}(16X_1 + 8X_2)}{\alpha} = 0 \\
\frac{\partial\theta}{\partial X_2} &= \frac{5\alpha^{1/2} - \beta\frac{1}{2}\alpha^{-1/2}(8X_1 + 12X_2 + 4X_3)}{\alpha} = 0 \\
\frac{\partial\theta}{\partial X_3} &= \frac{7\alpha^{1/2} - \beta\frac{1}{2}\alpha^{-1/2}(16X_3 + 4X_2)}{\alpha} = 0
\end{aligned}
$$

(n.b. factors of 10 not important)
Let $\beta X_i/\alpha^2 = z_i$ (using the usual trick) then
$3 \quad = 8z_1 + 4z_2 \qquad \Rightarrow z_1 = 3/8 - z_2/2$
$5 \quad = 4z_1 + 6z_2 + 2z_3$
$7 \quad = 2z_2 + 8z_3 \qquad \Rightarrow z_3 = 7/8 - z_2/4$
$\Rightarrow 5 - \frac{3}{2} = -2z_2 + 6z_2 + \frac{7}{4} - z_2/2 \qquad z + 2 = \frac{1}{2}$
Thence $z_1 = \frac{1}{8}, \quad z_3 = \frac{3}{4}$.
Now $\sum X_i = 1$

$$
\Rightarrow \frac{\beta}{\alpha^2} = \sum z_i = \frac{11}{8}
$$

Thus

$$
\begin{aligned}
X_1 &= \frac{1}{11} \\
X_2 &= \frac{4}{11} \\
X_3 &= \frac{6}{11}
\end{aligned}
$$

Thence $R_\Pi = \dfrac{1}{100}\left(\dfrac{8}{11} + \dfrac{40}{11} + \dfrac{77}{11}\right) = \dfrac{120}{11}\% \sim 10.91\%$

$$
\begin{aligned}
\frac{100^2}{10}\sigma_\Pi^2 &= \frac{8}{121} + \frac{32}{121} + \frac{96}{121} + \frac{8.36}{121} \\
&= \frac{520}{121} \\
&\Rightarrow \sigma_\Pi = \left(\sqrt{\frac{5200}{121}}\right)\frac{1}{100} \\
& \sigma_\Pi \sim 6.56\% \\
\theta &= \frac{\frac{120}{11} - \frac{500}{100}}{\frac{100}{100}\sqrt{\frac{5200}{121}}} \\
&= \sqrt{\frac{\sqrt{13}}{4}} \sim 0.90138
\end{aligned}
$$