QUESTION

(a) Obtain $\sin(2x)\cos(4x)$ as a sum of trigonometric functions, and hence evaluate

$$\int_0^{\frac{\pi}{2}} x\sin(2x)\cos(4x)\,dx.$$

(b) **(i)** Show that

$$\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x}.$$

(ii) Given that $x = 2$ is an approximate solution of the equation $2\tanh x = x$ use the Newton Raphson formula TWICE, and the result in part (i), to obtain a better approximation (correct to four decimal places).

ANSWER

(a) $\sin(2x)\cos(4x)$

From formula sheet, $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$
Therefore we require $\frac{A+B}{2} = 2x,\ \frac{A-B}{2} = 4x$ i.e.

$$\left.\begin{array}{r} A+B=4x \\ A-B=8x \end{array}\right\} \text{add: } 2A = 12x, A = 6x \Rightarrow B = -2x.$$

Therefore
$\sin 2x\cos 4x = \frac{1}{2}\{\sin(6x) + \sin(-2x)\} = \frac{1}{2}\{\sin(6x) - \sin(2x)\}$
Hence

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} x\sin 2x\cos 4x\,dx \\
= &\int_0^{\frac{\pi}{2}} \frac{x}{2}\{\sin 6x - \sin 2x\}\,dx \\
= &\frac{1}{2}\left\{\left[x\left(-\frac{\cos 6x}{6} + \frac{\cos 2x}{2}\right)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left(-\frac{\cos 6x}{6} + \frac{\cos 2x}{2}\right)dx\right\} \\
= &\frac{1}{2}\left\{\left[\frac{\pi}{2}\left(-\frac{\cos 3\pi}{6} + \frac{\cos\pi}{2}\right) - 0\right] + \left[\frac{\sin 6x}{36} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}\right\} \\
= &\frac{1}{2}\left\{\frac{\pi}{2}\left(\frac{1}{6} - \frac{1}{2}\right) + \frac{\sin 3\pi}{36} - \frac{\sin\pi}{4} - 0\right\} \\
= &\frac{\pi}{4}\left(-\frac{2}{6}\right) = -\frac{\pi}{12}
\end{aligned}$$

(b) (i) Now $u = \tanh x = \dfrac{\sinh x}{\cosh x}$, $\dfrac{du}{dx} = \dfrac{\cosh x \cosh x - \sinh x \sinh x}{(\cosh x)^2} = \dfrac{1}{\cosh^2 x}$

(ii) $f(x) = 2\tanh x - x$, $f'(x) = 2\text{sech}^2 x - 1$

given $x_0 = 2$,

$$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{(-0.071945)}{(-0.85870)} = 1.91622$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.91622 - \frac{(-0.00101)}{(-0.83401)} = 1.91501$$

Second approximation $\Rightarrow x = 1.9150$ (correct to 4 decimal places.)