Vector Calculus
Grad, Div and Curl Identities

Question
It is given that ϕ and ψ are scalar fields and $\underline{F}$ and $\underline{G}$ are vector fields. They are all assumed to be smooth functions. Prove the following identity

$$\nabla(\underline{f} \bullet \underline{G}) = \underline{F} \times (\nabla \times \underline{G}) + \underline{G} \times (\nabla \times \underline{F}) + (\underline{F} \bullet \nabla)\underline{G} + (\underline{G} \bullet \nabla)\underline{F}$$

Answer
The first component of $\nabla(\underline{F} \bullet \underline{G})$ is

$$\frac{\partial F_1}{\partial x}G_1 + F_1\frac{\partial G_1}{\partial x} + \frac{\partial F_2}{\partial x}G_2 + F_2\frac{\partial G_2}{\partial x} + \frac{\partial F_3}{\partial x}G_3 + F_3\frac{\partial G_3}{\partial x}.$$

We now need the calculate the first components of the terms on the right hand side of the equation.
For $\underline{F} \times (\nabla \times \underline{G})$, the first component is

$$F_2\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) - F_3\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right).$$

For $\underline{G} \times (\nabla \times \underline{F})$, the first component is

$$G_2\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - G_3\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right).$$

For $(\underline{F} \bullet \nabla)\underline{G}$, the first component is

$$F_1\frac{\partial G_1}{\partial x} + F_2\frac{\partial G_1}{\partial y} + F_3\frac{\partial G_1}{\partial z}.$$

For $(\underline{G} \bullet \nabla)\underline{F}$, the first component is

$$G_1\frac{\partial F_1}{\partial x} + G_2\frac{\partial F_1}{\partial y} + G_3\frac{\partial F_1}{\partial z}.$$

By adding these first components together it can be seen that all of the terms cancel out except those in the first component of $\nabla(\underline{F} \bullet \underline{G}$. Similar calculations for the other components yield similar results, hence

$$\nabla(\underline{f} \bullet \underline{G}) = \underline{F} \times (\nabla \times \underline{G}) + \underline{G} \times (\nabla \times \underline{F}) + (\underline{F} \bullet \nabla)\underline{G} + (\underline{G} \bullet \nabla)\underline{F}$$