Vector Calculus
Grad, Div and Curl Identities

Question
It is given that div$\underline{F} = 0$ in a domain D, of which any point P can be joined to the origin by a straight line segment in D. $\underline{r}$ is a parametrization of the line segment from the origin to (x, y, z) in D, with

$$\underline{r} = tx\underline{i} + ty\underline{j} + tz\underline{k}, \quad (0 \leq t \leq 1).$$

$\underline{G}$ is given by

$$\underline{G}(x, y, z) = \int_0^1 t\underline{F}(\underline{r}(t)) \times \frac{d\underline{r}}{dt}\, dt.$$

Show that curl$\underline{G} = \underline{F}$ throughout D.
Answer
Let $\underline{v} = x\underline{i} + y\underline{j} + z\underline{k}$. As the line segment lies inside D, so div$\underline{F} = 0$ on the path.
We have

$$\begin{aligned}
\underline{G}(x, y, z) &= \int_0^1 t\underline{F}(\underline{r}(t)) \times \underline{v}\, dt \\
&= \int_0^1 t\underline{F}(\xi(t), \eta(t), \zeta(t)) \times \underline{v}\, dt
\end{aligned}$$

With $\xi = tx$, $\eta = ty$ and $\zeta = tz$. So the first component of curl$\underline{G}$ is

$$\begin{aligned}
(\text{curl}\underline{G})_1 &= \int_0 1^1 t(\text{curl}(\underline{F} \times \underline{v}))_1\, dt \\
&= \int_0^1 t\left(\frac{\partial}{\partial y}(\underline{F} \times \underline{v})_3 - \frac{\partial}{\partial z}(\underline{F} \times \underline{v})_2\right) dt \\
&= \int_0^1 t\left(\frac{\partial}{\partial y}(F_1 y - F_2 x) - \frac{\partial}{\partial z}(F_3 x - F_1 z)\right) dt \\
&= \int_0^1 \left(tF_1 + t^2 y\frac{\partial F_1}{\partial \eta} - t^2 x\frac{\partial F_2}{\partial \eta} - t^2 x\frac{\partial F_3}{\partial \zeta}\right. \\
&\qquad \left. +tF_1 + t^2 z\frac{\partial F_1}{\partial \zeta}\right) dt \\
&= \int_0^1 \left(2tF - 1 + t^2 x\frac{\partial F_1}{\partial \xi} + t^2 y\frac{\partial F_1}{\partial \eta} + t^2 z\frac{\partial F_1}{\partial \zeta}\right) dt.
\end{aligned}$$

With the last line using the fact that div$\underline{F} = 0$ to replace $it^2 x\frac{\partial F_2}{\partial \eta} - t^2 x\frac{\partial F_3}{\partial \zeta}$ with $t^2 x\frac{\partial F_1}{\partial \xi}$.

So

$$
\begin{aligned}
(\text{curl}\underline{G})_1 &= \int_0^1 \frac{d}{dt}(t^2 F_1(\xi,\eta,\zeta))\,dy \\
&= t^2 F_1(tx,ty,tz)\Big|_0^1 \\
&= F_1(x,y,z)
\end{aligned}
$$

Similar arguements will show that $(\text{curl}\underline{G})_2 = F_2)$ and $(\text{curl}\underline{G})_3 = F_3$.

$$\Rightarrow \text{curl}\underline{G} - \underline{F}.$$