Vector Calculus

Grad, Div and Curl Identities

Question

It is given that $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, and that $\underline{F}$ is smooth.
Show that

$$\nabla \times (\underline{F} \times \underline{r}) = \underline{F} - (\nabla \bullet \underline{F})\underline{r} + \nabla(\underline{F} \bullet \underline{r}) - \underline{r} \times (\nabla \times \underline{F}).$$

In particular, if $\nabla \bullet \underline{F} = 0$ and $\nabla \times \underline{F} = \underline{0}$, then

$$\nabla \times (\underline{F} \times \underline{r}) = \underline{F} + \nabla(\underline{F} \bullet \underline{r}).$$

Answer

$$\begin{aligned} \nabla \times (\underline{F} \times \underline{r}) &= (\nabla \bullet \underline{r})\underline{F} + (\underline{r} \bullet \nabla)\underline{F} - (\nabla \bullet \underline{F})\underline{r} - (\underline{F} \bullet \nabla)\underline{r} \\ \nabla(\underline{F} \bullet \underline{r}) &= \underline{F} \times (\nabla \times \underline{r}) + \underline{r} \times (\nabla \times \underline{F}) + (\underline{F} \bullet \nabla)\underline{r} + (\underline{r} \bullet \nabla)\underline{F}. \end{aligned}$$

It can be seen that from the given $\underline{r}$, $\nabla \bullet \underline{r} = 3$ and $\nabla \times \underline{r} = \underline{0}$, and that

$$(\underline{F} \bullet \nabla)\underline{r} = F_1 \frac{\partial \underline{r}}{\partial x} + F_2 \frac{\partial \underline{r}}{\partial y} + F_3 \frac{\partial \underline{r}}{\partial z} = \underline{F}.$$

Using all of this leads us to

$$\begin{aligned} \nabla \times (\underline{F} \times \underline{r}) - \nabla(\underline{F} \bullet \underline{r} &= 3\underline{F} - 2(\underline{F} \bullet \nabla)\underline{r} - (\nabla \bullet \underline{F})\underline{r} \\ &\quad -\underline{r} \times (\nabla \times \underline{F}) \\ &= \underline{F} - (\nabla \bullet \underline{F})\underline{r} - \underline{r} \times (\nabla \times \underline{F}). \end{aligned}$$

If $\nabla \bullet \underline{F} = 0$ and $\nabla \bullet \underline{F} = \underline{0}$ then

$$\nabla \times (\underline{F} \times \underline{r}) - \nabla(\underline{F} \bullet \underline{r}) = \underline{F}.$$