

Vector Calculus
Grad, Div and Curl Identities

Question

It is given that $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, with $r = |\underline{r}|$. It is also given that f is a differentiable function of one variable.

Show that

$$\nabla \bullet (f(r)\underline{r}) = rf'(r) + 3f(r)$$

and find $f(r)$ if it is assumed that $f(r)\underline{r}$ is solenoidal for $r \neq 0$.

Answer

$$\begin{aligned}\nabla \bullet (f(r)\underline{r}) &= (\nabla f(r)) \bullet \underline{r} - f(r)(\nabla \bullet \underline{r}) \\ &= f'(r) \frac{\underline{r} \bullet \underline{r}}{r} + 3f(r) \\ &= rf'(r) + 3f(r)\end{aligned}$$

If $f(r)\underline{r}$ is solenoidal, then $\nabla \bullet (f(r)\underline{r}) = 0$, so that $u = f(r)$ will satisfy

$$\begin{aligned}r \frac{du}{dr} + 3u &= 0 \\ \frac{du}{u} &= -\frac{3 dr}{r} \\ \ln |u| &= -3 \ln |r| + \ln |C| \\ u &= Cr^{-3} \\ \Rightarrow f(r) &= Cr^{-3}\end{aligned}$$

for an arbitrary constant C .