QUESTION

(a) By considering $\int_0^K xe^{-x}\,dx$ determine whether or not the integral $\int_0^\infty xe^{-x}\,dx$ is defined, and if it is find its value.

(b) A double integral is defined by

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} y\,dx\,dy.$$

(i) Sketch the region of integration.

(ii) State the expression for an element of area in terms of plane polar coordinates (r, θ).

(iii) Rewrite the double integral I in terms of plane polar coordinates and hence evaluate the integral.

ANSWER

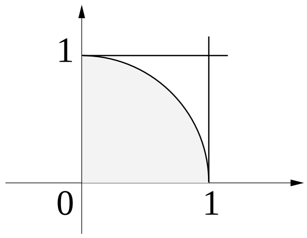
(a)

$$\begin{aligned}
\int_0^K xe^{-x}\,dx &= \left[\frac{xe^{-x}}{-1}\right]_0^K - \int_0^K \left(\frac{e^{-x}}{-1}\right).1\,dx \\
&= -\frac{K}{e^K} - 0 + \left[\frac{e^{-x}}{-1}\right]_0^K \\
&= -\frac{K}{e^K} - \frac{1}{e^K} + (e^{-0}) \\
&= 1 - \frac{K}{e^K} - \frac{1}{e^K}
\end{aligned}$$

$\int_0^\infty xe^{-x}\,dx = \lim_{K\to\infty}\left(\int_0^K xe^{-x}\,dx\right) = 1$ since both $\frac{1}{e^K}$ and $\frac{K}{e^K}$ tend to 0 as K tends to infinity.

(b) $I = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} y\,dx\,dy$

(i) $x = \sqrt{1-y^2}$, $x^2 = 1 - y^2$, i.e. $x^2 + y^2 = 1$ (a circle)



(ii) $dA = r dr d\theta$

(iii)

$$
\begin{aligned}
I &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} (r\sin\theta)(r\,dr\,d\theta) \\
&= \int_{0}^{\frac{\pi}{2}} \sin\theta\,d\theta \int_{0}^{1} r^2\,dr \\
&= [-\cos\theta]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^1 \\
&= \left(-\cos\frac{\pi}{2} - (-\cos 0)\right)\left(\frac{1}{3} - 0\right) \\
&= \frac{1}{3}
\end{aligned}
$$