**Question**

Use the mean value theorem to prove each of the following statements.

1. If $g'(x)$ is a polynomial of degree $n-1$, then $g(x)$ is a polynomial of degree $n$;

2. $x/(x+1) < \ln(1+x) < x$ for $-1 < x < 0$ and for $x > 0$;

3. $\sin(x) < x$ for $x > 0$;

**Answer**

1. Suppose that $g'(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$, and consider the new function $h(x) = \frac{1}{n}a_{n-1}x^n + \frac{1}{n-1}a_{n-2}x^{n-1} + \cdots + \frac{1}{2}a_1 x^2 + a_0 x - g(x)$. Note that since $g$ and polynomials are differentiable, and hence continuous, on all of $\mathbf{R}$, we have that $h$ is differentiable, and hence continuous, on all of $\mathbf{R}$. Also, $h'(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 - g'(x) = 0$ for all $x \in \mathbf{R}$.

   For $x_0 > 0$, apply the mean value theorem to $h$ on the interval $[0, x_0]$. Since $h$ is continuous on $[0, x_0]$ and differentiable on $(0, x_0)$, the mean value theorem yields that there exists some $c$ in $(0, x_0)$ so that $h(x_0) - h(0) = h'(c)(x_0 - 0) = 0$, since $h'(c) = 0$. That is, $h(x_0) = h(0)$ for all $x_0 > 0$. As above, we also get that $h(x_0) = h(0)$ for all $x_0 < 0$ by applying the mean value theorem to $h$ on the interval $[x_0, 0]$.

   Hence, setting $b = h(0)$, we have that $h(x) = b$ for all $x \in \mathbf{R}$. Substituting in the definition of $h$, this yields that $\frac{1}{n}a_{n-1}x^n + \frac{1}{n-1}a_{n-2}x^{n-1} + \cdots + \frac{1}{2}a_1 x^2 + a_0 x - g(x) = b$ for all $x \in \mathbf{R}$, that is, $g(x) = \frac{1}{n}a_{n-1}x^n + \frac{1}{n-1}a_{n-2}x^{n-1} + \cdots + \frac{1}{2}a_1 x^2 + a_0 x - b$ for all $x \in \mathbf{R}$, and so $g$ is a polynomial of degree $n$.

2. This is a slightly different sort of argument, and we break it into two pieces, corresponding to the two inequalities.

   Set $h(x) = x - \ln(x+1)$, and note that $h$ is differentiable, and hence continuous, on $(-1, \infty)$. The two cases, of $-1 < x < 0$ and of $x > 0$, are handled in the same fashion, and we write out the details only for the case $x > 0$. Apply the mean value theorem to $h$ on any closed interval in $[0, \infty)$. Note that $h(0) = 0 - \ln(1) = 0$. If there were another point $x_0 > 0$ at which $h(x_0) = 0$, then by applying either Rolle's theorem or the mean value theorem to $h$ on the interval $[0, x_0]$, there would exist a point $c$ in $(0, x_0)$ at which $h'(c) = 0$. However, $h'(c) = 1 - \frac{1}{c+1}$, which is non-zero for $c \neq 0$. Hence, $h(x) \neq 0$ for all $x \in (0, \infty)$. By the intermediate value theorem, this forces either $h(x) > 0$ for all $x > 0$ or $h(x) < 0$ for all $x > 0$ (because if there are points $a$ and $b$ in $(0, \infty)$ at which $h(a) > 0$ and $h(b) < 0$, then there is a point $c$ between $a$ and $b$ at which $h(c) = 0$). Since $h(1) = 1 - \ln(2) = 0.3069... > 0$, we have that $h(x) > 0$ on $(0, \infty)$, that is, that $x > \ln(x+1)$ for all $x > 0$, as desired. (As noted above, the argument to show that $h(x) > 0$ for $-1 < x < 0$, or equivalently that $x > \ln(x+1)$ for $-1 < x < 0$, is similar, and is left for you to write out.)

For the other inequality, set $g(x) = \ln(x+1) - \frac{x}{x+1}$, and note that $g$ is differentiable, and hence continuous, for $x > -1$. (As above, we give the details in the case that $x > 0$, and leave the case of $-1 < x < 0$ to you the reader.) Note that $g'(x) = \frac{x}{(x+1)^2} > 0$ for $x > 0$. In particular, applying the mean value theorem to $g$ on the interval $[0, x_0]$, we see that there is $c$ in $(0, x_0)$ so that $g(x_0) - g(0) = g'(c)(x_0 - 0) > 0$, since both $g'(c) > 0$ and $x_0 > 0$. Hence, $g(x_0) > g(0) = 0$ for all $x > 0$. That is, $\ln(x+1) > \frac{x}{x+1}$ for all $x > 0$.

3. Here, set $g(x) = x - \sin(x)$. We wish to show that $g(x) > 0$ for all $x > 0$. First, note that since $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbf{R}$, we have that $g(x) > 0$ for $x > 1$, and so we can restrict our attention henceforth to $0 < x \leq 1$. Also, note that $g(x)$ is differentiable, and hence continuous, on all of $\mathbf{R}$, and so we may apply the mean value theorem to $g$ on any closed interval $[0, x_0]$ for $0 < x_0 \leq 1$. So, there exists some $c$ in $(0, x_0)$ so that $g(x_0) - g(0) = g'(c)(x_0 - 0)$. Since $g(0) = 0$ and since $g'(c) = 1 - \cos(c) > 1$ for $c \in (0, 1)$, we have that $g(x_0) > 0$ for all $0 < x_0 \leq 1$, and hence that $g(x) > 0$ for all $x > 0$, as desired.