Question

a) Give the Laurent series expansions in powers of z for

$$f(z) = \frac{1}{z^2(z-2)}$$

in each of the two regions $0 < |z| < 2$ and $|z| > 2$.

Hence, or otherwise, evaluate $\int_{C_r} f(z)dz$, where C_r is the circle $|z| = r$ in the clockwise sense in each of the cases $0 < r < 2$ and $2 < r$.

b) Use the calculus of residues to show that

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

Answer

a) For $0 < |z| < 2$

$$\frac{1}{z^2(z-2)} = \frac{1}{z^2}\left(\frac{-1}{2\left(1-\frac{z}{2}\right)}\right) = \frac{-1}{2z^2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$= \frac{-1}{2z^2} - \frac{1}{4z} - \frac{1}{8} - \frac{z}{16} - \cdots - \frac{z^n}{2^{n+3}} - \cdots$$

For $|z| > 2$

$$\frac{1}{z^2(z-2)} = \frac{1}{z^3\left(1-\frac{2}{z}\right)} = \frac{1}{z^3}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right)$$

$$= \frac{1}{z^3} + \frac{2}{z^4} + 4z^5 + \cdots + \frac{2^{n-3}}{z^n} + \cdots$$

$\int_{C_r} f(z)dz = 2\pi i \sum$residues within C_r

The residue at $z = 2$ is $\lim_{z\to 2} z - 2f(z) = \frac{1}{4}$

The residue at $z = 0$ is $-\frac{1}{4}$ from the first expansion.

So if $r < 2$, $\oint_{C_r} f(z)dz = -2\pi i \frac{1}{4} = -\frac{\pi}{2}$ (going clockwise)

Thus $\oint_{C_r} f(z)dz = \frac{\pi i}{2}$ (going anticlockwise)

In the case $r > 2$ the sum of the residues is zero,

So $\int_{C_r} f(z)dz = 0$ in both directions.

b) Let $f(z)\frac{1}{(z^2+1)^2}$, now $\frac{1}{(x^2+1)^2} \leq \frac{1}{x^4}$ and $\int^{\infty} \frac{1}{x^4}$ converges.

Integrate $f(z)$ round Γ, with $R > 1$

$f(z)$ has a pole of order 2 at $z = i$ inside Γ, with residue $-\frac{1}{4}$ using the diffn formula.

Thus $\int_{\Gamma} f(z)dz = \frac{\pi}{2}$

$\left| \int_{\text{semicircle}} f(z)dz \right| \leq \frac{\pi R}{(R^2-1)^2} \to 0$ as $R \to \infty$

Thus $\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}$ and so $\int_0^{\infty} \frac{1}{(x^2+1)^2} dx = \frac{\pi}{4}$