Question

a) Show that if $f(z)$ has a pole of order 2 at $z = a$, then its residue at $z = a$ is

$$\lim_{z\to a}\left(\frac{d}{dz}((z-a)^2 f(z))\right).$$

b) The function $f(z)$ is regular except for $z = 0$ and $z = i$. Find $f(z)$ when

i) $f(z)$ has a simple pole at $z = 0$,

ii) $f(z)$ has a pole of order 2 with residue 5 at $z = i$,

iii) $\lim_{z\to\infty} zf(z) = 3$ and

iv) $f(-i) = 0$.

Answer

a) Bookwork from the Laurent Series

b) We look for a function of the form

$f(z) = \dfrac{P(z)}{z(z-i)^2}$ using (i) and (ii).

since $\lim_{z\to\infty} zf(z)$ is finite and non zero, P is quadratic.

$\lim_{z\to\infty} \dfrac{z(\alpha z^2 + \beta z + \gamma)}{z(z-i)^2} = 3$, therefore $\alpha = 3$.

The residue at $z = i$ is

$\lim_{z\to i} \dfrac{d}{dz}\dfrac{3z^2+\beta z+\gamma}{z} = \lim_{z\to i} 3 - \dfrac{\gamma}{z^2} = 3 + \gamma = 5$ so $\gamma = 2$

$f(-i) = 0$ so $3(-i)^2 + \beta(-i) + 2 = 0$

$\beta(-i) - 1 = 0$ so $\beta = i$

Thus $f(z)\dfrac{3z^2+iz+2}{z(z-1)^2}$