Question

a) Let x, y be the real and imaginary parts of the complex number z and let $w = \tan z$. By finding the modulus and argument of $\frac{w-i}{w+i}$ in terms of x, y show that a line y=constant of the z-plane transforms to a circle of the w-plane, and that a line x=constant transforms to an arc of a circle through i and $-i$.

b) Find all solutions of $\tan z = 2 - i$.

Answer

a) $w = \tan z = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = -i\dfrac{e^{2iz} - 1}{e^{2iz} + 1}$

so $e^{2iz}w + w + ie^{2iz} - i = 0 \qquad e^{2iz} = -\dfrac{w-i}{w+i}$

So $\dfrac{w-i}{w+i} = -e^{-2iz} = -e^{-2ix}e^{2y} = e^{-2ix+i\pi}e^{2y}$

So $\left|\dfrac{w-i}{w+i}\right| = e^{2y} \qquad \arg\left(\dfrac{w-i}{w+i}\right) = \pi - 2x$

So if y=constant $\qquad \left|\dfrac{w-i}{w+i}\right|$=constant - circle of Apallonius.

If x=constant $\qquad \arg\left(\dfrac{w-i}{w+i}\right)$=constant - circular arc.

DIAGRAM

b) $\tan z = 2 - i$, so $e^{2iz} = -\dfrac{2-i-i}{2-i+i} = -1 + i = \sqrt{2}e^{\frac{3\pi i}{4}} = \alpha$

$$
\begin{aligned}
2iz &= \ln|\alpha| + i(\arg\alpha + 2n\pi) \\
&= \frac{1}{2}\ln 2 + i\left(\frac{3\pi}{4} + 2n\pi\right)
\end{aligned}
$$

$$z = \frac{1}{4i}\ln 2 + \frac{3\pi}{8} + n\pi \qquad n \in \mathbf{Z}$$