Question

a) Use the Cauchy-Riemann equations to prove that the regular function $f(z)$, with $z = x + iy$, satisfies

$$\left(\frac{\partial|f|}{\partial x}\right)^2 + \left(\frac{\partial|f|}{\partial y}\right)^2 = |f'(z)|^2$$

when $f(z) \neq 0$.

b) Find a regular function of $z = x + iy$ with real part $e^y \cos x$.

Answer

a) $|f|^2 = u^2 + v^2$

So $2|f|\dfrac{\partial|f|}{\partial x} = 2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x}$

and $2|f|\dfrac{\partial|f|}{\partial y} = 2u\dfrac{\partial u}{\partial y} + 2v\dfrac{\partial v}{\partial y} = -2u\dfrac{\partial v}{\partial x} + 2v\dfrac{\partial u}{\partial x}$

$$|f|^2\left(\left(\frac{\partial|f|}{\partial x}\right)^2 + \left(\frac{\partial|f|}{\partial y}\right)^2\right) = (u^2+v^2)\left(\frac{\partial u}{\partial x}\right)^2 + (u^2+v^2)\left(\frac{\partial v}{\partial x}\right)^2$$

$$= |f|^2\left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f|^2|f'(z)|^2$$

So $\left(\dfrac{\partial|f|}{\partial x}\right)^2 + \left(\dfrac{\partial|f|}{\partial y}\right)^2 = |f'(z)|^2$ if $|f| \neq 0$

b) $u = e^y \cos x$

$\dfrac{\partial u}{\partial x} = -e^y \sin x = \dfrac{\partial v}{\partial y} \Rightarrow v = -e^y \sin x + \phi(x)$

$-\dfrac{\partial u}{\partial y} = -e^y \cos x = \dfrac{\partial v}{\partial x} \Rightarrow v = -e^y \sin x + \psi(y)$

so $v = -e^y \sin x$ will do

$f = u + iv = e^y(\cos x - i\sin x) = e^{y-ix} = e^{i(x+iy)} = e^{iz}$