Question

Suppose that $X \sim \text{beta}(\alpha, \beta)$. Show that the pdf of $Y = \frac{1}{X} - 1$ is

$$g(y|\alpha,\beta) = \begin{cases} \frac{1}{B(\alpha,\beta)} \frac{y^{\beta-1}}{(1+y)^{\alpha+\beta}}, & \text{for } y \geq 0; \\ 0, & \text{otherwise} \end{cases}$$

This distribution is called the **beta distribution of the second type**.

Answer

We have $f(x|\alpha,\beta)\dfrac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$

The transformation $Y = \dfrac{1}{X} - 1$ is decreasing and continuous if $x \in (0,1)$ and also $0 < y < \infty$.

$$x = \frac{1}{1+y} \Rightarrow \frac{dx}{dy} = -\frac{1}{(1+y)^2}$$

Therefore the pdf of Y is

$$\begin{aligned} g(y) &= \frac{1}{B(\alpha,\beta)} \cdot \left(\frac{1}{1+y}\right)^{\alpha-1} \cdot \left(1 - \frac{1}{1+y}\right)^{\beta-1} \cdot \left(\frac{1}{1+y}\right)^2, \quad 0 \leq y < \infty \\ &= \frac{1}{B(\alpha,\beta)} \cdot \frac{y^{\beta-1}}{(1+y)^{\alpha+\beta}}, \quad 0 \leq y < \infty \end{aligned}$$