Question

If $X \sim \text{exponential}(\beta)$ then find the pdf of $Y = X^{\frac{1}{\gamma}}$. The random variable Y is known as the Weibull random variable. Using a list of distributions, write down its mean and variance.

Answer

Here $f(x|\beta) = \frac{1}{\beta} e^{\frac{-x}{\beta}}, \quad 0 < x < \infty$

$$Y = X^{\frac{1}{\gamma}} \quad \gamma > 0$$

The transformation is increasing and continuous.

Therefore $x = y^\gamma \Rightarrow \frac{dx}{dy} = \gamma y^{\gamma-1}$

Therefore the pdf of

$$\begin{aligned} Y = g(y) &= \frac{1}{\beta} \cdot e^{-\frac{y^\gamma}{\beta}} \cdot \left| \gamma y^{\gamma-1} \right|, \quad 0 < y < \infty \\ &= \frac{\gamma}{\beta} \cdot y^{\gamma-1} \cdot e^{-\frac{y^\gamma}{\beta}}, \quad 0 < y < \infty. \end{aligned}$$

$$E(Y) = \beta^{\frac{1}{\gamma}} \Gamma\left(1 + \frac{1}{\gamma}\right) \quad \text{and } \operatorname{var}(Y) = \beta^{\frac{2}{\gamma}} \left\{ \Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right) \right\}$$