Question

Prove that multiplication of nxn matrices is associative, by verifying that the ij-th element of $(AB)C$ and $A(BC)$ have the common value

$$\sum_{p,q=1,\cdots,n} a_{1p}b_{pq}c_{qj}.$$

(try with the ij notation of 2x2 and 3x3 matrices first if you are not sure what to do for nxn.)

Answer

$$\begin{aligned}
(AB)C_{ij} &= \text{ith row of } AB \cdot \text{jth column of } C \\
&= AB_{i1}C_{1j} + AB_{i2}C_{2j} + \cdots AB_{in}C_{nj} \\
&= \sum_{q=1}^{n} (AB)_{iq}c_{qi} \\
&= \sum_{q=1}^{n} (\text{ith row of } A \cdot \text{qthcolumn of } B)c_{qj} \\
&= \sum_{q=1}^{n} \left(\sum_{p=1}^{n} a_{ip}b_{pq} \right) c_{qj} \\
&= \sum_{p,q=1}^{n} a_{ip}b_{pq}c_{qj} \\
A(BC)_{ij} &= \text{ith row of } A \cdot \text{jth column of } BC \\
&= \sum_{p=1}^{n} a_{ip}BC_{pj} \\
&= \sum_{p=1}^{n} a_{ip} \sum_{q=1}^{n} b_{pq}c_{qj} \\
&= \sum_{p,q=1}^{n} a_{ip}b_{pq}c_{qj}
\end{aligned}$$