Question

Prove that

$$\begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix} = \cos 3\theta.$$

Evaluate

$$\begin{vmatrix} 2\cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix}.$$

For which values of θ are the corresponding 3x3 matrices singular. Write down the inverse when they exist.

Answer

$$\begin{aligned} A &= \begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix} \\ &= \cos\theta(4\cos^2\theta - 1) - 2\cos\theta \\ &= 4cos^3\theta - 3\cos\theta \\ &= \cos 3\theta \end{aligned}$$

$$\begin{aligned} B &= \begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix} \\ &= 2\cos\theta(4\cos^2\theta - 1) - \cos\theta \\ &= 2\cos\theta(4\cos^2 - 2) \\ &= 4\cos\theta\cos 2\theta \\ &= \frac{\sin 4\theta}{\sin\theta} \quad (\sin\theta \neq 0) \end{aligned}$$

The matrix of A is singular if $\cos 3\theta = 0$ so $\theta = (2n+1)\frac{\pi}{6}$
The matrix B is singular if $\cos\theta = 0$ or $\cos 2\theta = 0$ So $\theta = (2n+1)\frac{\pi}{2}$ and $\theta = (2n+1)\frac{\pi}{4}$.

The inverse of A is

$$\frac{1}{\cos 3\theta}\begin{pmatrix} 4\cos^2\theta - 1 & -2\cos\theta & 1 \\ -2\cos\theta & 2\cos^2\theta & -\cos\theta \\ 1 & -\cos\theta & 2\cos^2\theta - 1 \end{pmatrix}$$

The inverse of B is

$$\frac{1}{4\cos\theta\cos 2\theta}\begin{pmatrix} 4\cos^2\theta - 1 & -2\cos\theta & 1 \\ -2\cos\theta & 4\cos^2\theta & -2\cos\theta \\ 1 & -2\cos\theta & 4\cos^2\theta - 1 \end{pmatrix}$$