QUESTION

Find all possible simultaneous solutions to the following sets of congruences, expressing your answers as congruence classes modulo some suitable integer.

(i) $x \equiv 2 \bmod 7$.

$x \equiv 7 \bmod 9$.

$x \equiv 3 \bmod 4$.

(ii) $x^2 + 2x + 2 \equiv 0 \bmod 5$.

$7x \equiv 3 \bmod 11$.

ANSWER

(i) 7,9 and 4 are mutually coprime, so the Chinese Remainder Theorem guarantees a solution, which is unique mod $7.9.4 = 252$, You may have followed the method of the Chinese Remainder Theorem, or gone for the quick method. Here are solutions for both:-

CHINESE REMAINDER THEOREM

Here $n = 7.9.4 - 252$, $N_1 = \frac{n}{7} = 36$, $N_2 = \frac{n}{9} = 28$ and $N_3 = \frac{n}{4} = 63$. We must solve $36x_1 \equiv 1 \bmod 7$, $28x_2 \equiv 1 \bmod 9$ and $63x_3 \equiv 1 \bmod 4$. These simplify to $x_1 \equiv 1 \bmod 7$, $x_2 \equiv 1 \bmod 9$ and $-x_3 \equiv 1 \bmod 4$, so we may take $x_1 = 1, x_2 = 1$ and $x_3 = 3$. The Chinese Remainder Theorem then tells us that $\overline{x} = 2.36.1+7.28.1+3.63.3$ is a simultaneous solution. Now $\overline{x} = 72 + 196 + 567 = 835 \equiv 79 \bmod 252$ so our solution is $x \equiv 79 \bmod 252$.

QUICK METHOD

The Chinese Remainder Theorem guarantees a congruence class of solutions mod 252, so guarantees integer solutions bigger than any preordained size.

We start with the equation of largest modulus, $x \equiv 7 \bmod 9$, find an integer soluytion (7), then increase it by multiples of 9 until we reach a solution of the next congruence $x \equiv 2 \bmod 7$, viz. 7,16.

16 is a common solution of $x \equiv 7 \bmod 9$ and $x \equiv 2 \bmod 7$. We increase this by multiples of 9.7 (so that the numbers on our list are solutions to both equations), until we reach a solution of the final equation, $x \equiv 3 \bmod 4$, viz. 16, 79.

Thus $x \equiv 79 \bmod 252$ simultaneously solves all three equations.

(ii) We begin by solving the congruences:

For $x^2+2x+2 \equiv 0 \bmod 5$ we have not yet learnt a general method (see §7), but as 5 is small , we may try out all congruence classes mod 5, and pick out the solutions. The least absolute residues mod 5 are $0, \pm 1, \pm 2$, and we see that $f(0) \equiv 2 \bmod 5$,$f(1) = 5 \equiv 0 \bmod 5$, $f(-1) \equiv 1 \bmod 5$, $f(2) = 10 \equiv 0 \bmod 5$ and $f(-2) \equiv 2 \bmod 5$, so the solutions of the congruence are $x \equiv 1 \bmod 5$ and $x \equiv 2 \bmod 5$.

To solve $7x \equiv 3 \bmod 11$, we could use, for example, $7x \equiv 3 \equiv 14 \bmod 11$, so on division by 2, $x \equiv 2 \bmod 11$.

Thus a simultaneous solution of both congruences would satisfy either $x \equiv 1 \bmod 5$ and $x \equiv 2 \bmod 11$ or $x \equiv 2 \bmod 5$ and $x \equiv 2 \bmod 11$.

The Chinese Remainder Theorem guarantees a unique solution for eac pair of equations mod 55, so we will end up with two congruence classes mod 55 as solutions. Again we have a choice of two methods- this time I'll use the quick method:-

For $x \equiv 1 \bmod 5$ and $x \equiv 2 \bmod 22$, start with a solution (2) for $x \equiv 2 \bmod 11$, and increase by multiples of 11 until we reach a solution of $x \equiv 1 \bmod 5$.

We get 2,13,24,35,46, so a suitable solution is $x \equiv 46 \bmod 55$.

For $x \equiv 2 \bmod 5$ and $x \equiv 2 \bmod 11$, we immediately see (as the solution is unique mod 55) that $x \equiv 2$ is the answer.

Thus the two congruences are solved by either $x \equiv 2$ or $x \equiv 46 \bmod 55$.