QUESTION

For each of the following congruences, decide whether or not a solution exists. If a solution does exist, find all possible solutions, describing them as congruence classes.

(i) $3x \equiv 5 \mod 7$.

(ii) $12x \equiv 15 \mod 22$.

(iii) $19x \equiv 42 \mod 50$.

(iv) $18x \equiv 42 \mod 50$.

ANSWER

(i) gcd93,7)=1, which divides 5, so by theorem 3.5 end corollary 3.6, this congruence has solutions, which form a single congruence class mod 7.

Multiplying the congruence through by 2 gives $6x \equiv 10 \equiv 3 \mod 7$, that is $-x \equiv 3 \mod 7$, so that $x \equiv -3 \equiv 4 \mod 7$.

(Alternatively, use $3x \equiv 5 \equiv 12 \mod 7$, and then divide through by 3.)

(ii) gcd(12,22)=2 which does not divide 15, so by theorem 3.5 this equation has no solutions.

(iii) gcd(12,22)=1, which divides 42, so solutions exist, and by cor.6.3 they form a single congruence class mod 50.

Multiplying the congruence through by 3 gives $57x \equiv 3.42 \mod 50$, that is $7x \equiv 3.42 \mod 50$, which upon division by 7 yields $x \equiv 3.6 \equiv 18 \mod 50$. (Again, alternative methods exist- maybe you found a quicker one?)

(iv) gcd(18,50)=2, which divides 42, so by theorem 3.5, solutions exist, and fall into two congruence classes mod 50. If x_0 gives on class of solutions, $x_0 + \frac{50}{2} = x_0 + 25$ gives the other. Thus we need only one integer x_0 satisfying the congruence. The two congruence classes of solutions will be those with representatives x_0 and $x_0 + 25$. If $18x \equiv 42 \mod 50$, then dividing the whole equation (including the modulus) through by 2 (see lemma 3.4(i)) gives $9x \equiv 21 \mod 25$. Multiplying by 3 then gives $27x \equiv 21.3 \equiv (-4).3 \equiv -12 \mod 25$, which gives $2x \equiv -12 \mod 25$. We may now divide by 2 to get $x \equiv -6 \equiv 19 \mod 25$. Thus the solutions of the original equation are $x \equiv 19 \mod 50$ and $x \equiv 44 \mod 50$.