QUESTION
Let $f(x) = x^3 - x + 1$
Calculate the values of $f(0),\ f(1),\ f(2)$ modulo 3, and hence deduce that $f(x) = 0$ has no integer solutions.
ANSWER
$f(0) = 0 - 0 + 1 = 1,\ f(1) = 1 - 1 + 1 = 1$ and $f(2) = 8 - 2 + 1 = 7$, and so $f(0), f(1)$ and $f(2)$ are all congruent to 1 modulo 3. If n were an integer solution of $f(x) = 0$, we'd have $f(n) = 0$, and hence $f(n) \equiv 0 \bmod 3$. But n must be congruent to one of 0,1,2 mod 3, by lemma 3.3, and so by the above this gives $f(n) \equiv 1 \bmod 3$. This contradiction shows that we cannot have any integer solutions of $f(x) = 0$.