Question

Two dealers in financial derivatives set out to make millions. Both start with \$1000. The first is very careful with his investments and manages to increase his wealth at a rate proportional to the amount of money he has. The second dealer is much more speculative in her approach and manages to increase her wealth at a rate proportional to the square of the amount of money she has. At the end of the first year of trading both dealers, although they have used different strategies, have each managed to be worth \$2000. If they both continue with their own strategies how much will they each be worth at the end of the third year? (please explain any unusual behaviour in your answers). (*)

Answer

Analysis for Dealer 1. Take $Q(t)$ = money measured in \$.

Equation for money is $\dfrac{dQ}{dt} = kQ$ with $Q(0) = 1000$ and $Q(1) = 2000$.

Solving the equation give $Q = Ae^{kt}$

Applying conditions give $A = 1000$ and $2000 = 1000e^k$

Hence $k = \ln 2$

$q(t) = 1000e^{t\ln 2} = 1000(2)^t$

Hence at $t = 3$ Q(3)=\$8000.

Analysis for Dealer 2.

Equation for money is $\dfrac{dQ}{dt} = KQ^2$ with $Q(0) = 1000$ and $Q(1) = 2000$.

The equation is separable so that

$$\int \frac{dQ}{Q^2} = K\int dt \qquad -\frac{1}{Q} = Kt + c \qquad Q = \frac{1}{-C - Kt}$$

Using the conditions implies $1000 = \frac{1}{-C}$ hence $C = \frac{-1}{1000}$

Also $2000 = \dfrac{1}{\frac{1}{1000} - K}$ implying $K = \frac{1}{2000}$

So that $Q(t) = \dfrac{2000}{2 - t}$

Hence at $t = 3$ we have Q(3)=-\$2000

The negative amount of money occurs because, although $Q(t)$ appears to be a monotonic increasing function $\left(\dfrac{dQ}{dt} = Q^2 > 0\right)$ the function becomes unbounded at $t = 2$ and the model is not physically valid beyond that time.