Question

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ be vectors with $\mathbf{a}\cdot\mathbf{b}\neq 0$, and let t be a scalar. Show that the equation:

$$\mathbf{a}\times\mathbf{r}=\mathbf{a}+t\mathbf{b}$$

can be satisfied only if

$$t=-\frac{\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}}.$$

Put this value of t into the equation and deduce that $\mathbf{r}$ must have the form:

$$\mathbf{r}=\frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}}+\left(\frac{\mathbf{a}\cdot\mathbf{r}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}.$$

Answer

$\mathbf{a}\times\mathbf{r}=\mathbf{a}+t\mathbf{b}\qquad(*)$

Take the dot products of both sides of (*) with $\mathbf{a}$:

$(\mathbf{a}\times\mathbf{r})\cdot\mathbf{a}=(\mathbf{a}+t\mathbf{b})\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{a}+t\mathbf{b}\cdot\mathbf{a}$

and since $(\mathbf{a}\times\mathbf{r})\cdot\mathbf{a}=0$ we have $0=\mathbf{a}\cdot\mathbf{a}+t\mathbf{b}\cdot\mathbf{a}$ or $\mathbf{a}\cdot\mathbf{b}t=-\mathbf{a}\cdot\mathbf{a}$

Dividing by the scalar $\mathbf{a}\cdot\mathbf{b}$ gives $t=\dfrac{-\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}}$

Substitute this value of t into (*) to obtain:

$$\mathbf{a}\times\mathbf{r}=\mathbf{a}+\left(\frac{-\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}\qquad(**)$$

Take the cross product of both sides of (**) with $\mathbf{a}$:

$$\begin{aligned}\mathbf{a}\times(\mathbf{a}\times\mathbf{r}) &= \mathbf{a}\times\left(\mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}\right)\\ &= \mathbf{a}\times\mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}}\right)(\mathbf{a}\times\mathbf{b})\quad\text{(Distributive Property)}\\ &= -\left(\frac{\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}}\right)(\mathbf{a}\times\mathbf{b})\quad(\text{since } \mathbf{a}\times\mathbf{a}=0)\end{aligned}$$

From question 3 we have:

$$\begin{aligned}\mathbf{a}\times(\mathbf{a}\times\mathbf{r}) &= (\mathbf{a}\cdot\mathbf{r})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{r}\\ \text{and so}\quad(\mathbf{a}\cdot\mathbf{r})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{r} &= -\left(\frac{\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}}\right)(\mathbf{a}\times\mathbf{b}).\\ \text{Rearranging gives}\quad\mathbf{r} &= \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}}+\left(\frac{\mathbf{a}\cdot\mathbf{r}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}\\ (\text{assume } \mathbf{a} \text{ is a non}-\text{zero vector, so}\ \ \mathbf{a}\cdot\mathbf{a} &= |\mathbf{a}|^2\neq 0)\end{aligned}$$