Question

a) Evaluate the sum of the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2}$$

You should justify the steps in your method, except that you may assume without proof inequalities relating to the function $\cot \pi z$.

b) State Rouche's theorem and use it to show that all the roots of the equation

$$z^7 + (1-i)z^5 + 2z^3 - 1 = 0$$

lie in the annulus $\frac{1}{2} \leq |z| < 2$. Find a value of k smaller than 2 with the property that all the roots of the equation satisfy $|z| < k$.

Answer

a) $S = \sum_{-\infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{4} \sum_{n=-\infty}^{\infty} \frac{1}{\left(n-\frac{1}{2}\right)^2}$

The function $f(z) = \frac{\pi \cot \pi z}{\left(z-\frac{1}{2}\right)^2}$ has a simple pole at $z = n \in \mathbf{N}$ with residue $\frac{1}{\left(n-\frac{1}{2}\right)^2}$,

since $(z-n)f(z) = \frac{\pi(z-n)}{\sin \pi(z-n)} \frac{\cos \pi(z-n)}{\left(z-\frac{1}{2}\right)^2} \to \frac{1}{\left(n-\frac{1}{2}\right)^2}$ as $z \to n$.

$f(z)$ has a pole of order 2 at $z = \frac{1}{2}$. The residue is given by

$$\lim_{z\to\frac{1}{2}} \frac{d}{dz}\left(\left(z-\frac{1}{2}\right)^2 f(z)\right) = \lim_{z\to\frac{1}{2}} \frac{d}{dz}\pi \cot \pi z$$

$$= -\pi^2 \csc^2 \frac{1}{2}\pi = -\pi^2$$

Now let C_N be the square with vertices $\pm(N+\frac{1}{2})(1 \pm i)$ $N \geq 0$.

On C_N $\pi \cot \pi z$ is uniformly bounded (by K).

So $\left| \int_{C_N} f(z)dz \right| \leq \dfrac{K8\left(N+\frac{1}{2}\right)}{N^2} \to 0$ as $N \to \infty$.

But $\displaystyle\int_{C_N} f(z)dz = 2\pi i \left(\sum_{-N}^{N} \frac{1}{\left(n-\frac{1}{2}\right)^2} - \pi^2 \right)$

so letting $N \to \infty$, $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{\left(n-\frac{1}{2}\right)^2} = \pi^2$. Thus $S = \frac{\pi^2}{4}$.

b) Rouche's Theorem states that if $f(z)$ and $g(z)$ are both analytic inside and on the closed contour C, and if $|g(z)| < |f(z)|$ on C then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C

i) Let $f(z) = -1$, $g(z) = z^7 + (1-i)z^5 + 2z^3$,
for $|z| = \frac{1}{2}$ $|g(z)| \leq (\frac{1}{2})^7 + \sqrt{2}(\frac{1}{2})^5 + 2(\frac{1}{2})^3 < 1 = |f(z)|$
$f(z)$ has no zeros inside $|z| = \frac{1}{2}$, so $f(z)+g(z)$ has none inside $|z| = \frac{1}{2}$.

ii) Let $f(z) = z^7$, $g(z) = (1-i)z^5 + 2z^3 - 1$,
for $|z| = 2$ $|g(z)| \leq \sqrt{2}2^5 + 2^4 + 1 < 2^6 + 2^4 + 1 = 81 < 2^7 = |f(z)|$
$f(z)$ has 7 zeros inside $|z| = 2$, and so does $f(z)+g(z)$ therefore.
For $|z| = 1.6$ $|g(z)| < \sqrt{2}(1.6)^5 + 2(1.6)^4 + 1 \approx 24.02$
$|f(z)| = (1.6)^7 \approx 26.84$ So $a = 1.6$ will do.
(This doesn't quite work with $a = 1.5$).