Question

a) Find the Taylor series expansion of the function $f(z) = (z+1)/(z-1)$ about the origin.

Find the Laurent expansion of $f(z)$ about the origin for $|z| > 1$.

Find the Laurent expansion of $f(z)$ about $z = 1$.

b) Locate the zeros and singularities of the function

$$\frac{z^2(z^2 - z + 1)\exp(1/z)}{z^3 - 13z^2 + 5z + 7}$$

Classify the singularities, and determine the behaviour of the function at infinity.

Answer

a) For $|z| < 1$, $\dfrac{1}{1-z} = 1 + z + z^2 + \cdots$

So $\dfrac{z+1}{z-1} = -(1+z)(1+z+z^2+\cdots) = -(1+2z+2z^2+2z^3+\cdots)$, this is the required Taylor expansion

For $|z| > 1$ $\dfrac{z+1}{z-1} = \dfrac{z+1}{z\left(1-\frac{1}{z}\right)} = \left(1+\dfrac{1}{z}\right)\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\cdots\right)$

$= 1 + \dfrac{2}{z} + \dfrac{2}{z^2} + \dfrac{2}{z^3} + \cdots$ this is the required Laurent expansion about the origin.

Now $\dfrac{z+1}{z-1} = \dfrac{z-1+2}{z-1} = 1 + \dfrac{2}{z-1}$, this is the expansion about $z = 1$, having only two terms.

b) $z^2 - z + 1 = 0$ when $z = \frac{1 \pm i\sqrt{3}}{2}$. These are zeros of the function.

$z^3 - 13z^2 + 5z + 7 = (z-1)(z^2 - 12z - 7) = 0$ when $z = 1$ and $z = 6 \pm \sqrt{43}$.

So the function has simple poles at $z = 1$ and $z = 6 \pm \sqrt{43}$.

The function has an essential singularity at $z = 0$.

To investigate the behaviour at infinity, replace z by $\frac{1}{z}$ to obtain.

$$\frac{\frac{1}{z^2}\left(\frac{1}{z^2}-\frac{1}{z}+1\right)\exp(z)}{\frac{1}{z^3}-\frac{13}{z^2}+\frac{5}{z}+7}=\frac{(1-z+z^2)e^z}{z(1-13z+5z^2+7z^3)}$$

This has a simple pole at $z=0$, so the original function has a simple pole at infinity.