QUESTION

Find the principal part of the Laurent expansions of the following functions at the point a.

(a) $\frac{e^z}{(z-2)^2},\ a=2$

(b) $\frac{\sin z}{z^3(z+1)^2},\ a=0$

ANSWER

(a)

$$\begin{aligned}\frac{e^z}{(z-2)^2} &= (z-2)^{-2}\, e^2 e^{z-2} \\ &= e^2 \sum_{n=0}^{\infty} \frac{(z-2)^{n-2}}{n!} \\ &= \sum_{m=-2}^{\infty} \frac{e^2}{(m+2)!}\,(z-2)^m \text{ (Taking } m=n-2)\end{aligned}$$

The principal part is

$e^2\,(z-2)^{-2} + e^2\,(z-2)^{-1}$

(b)

$$\begin{aligned}\frac{\sin z}{z^3\,(z+1)^2} &= z^{-3}\left(z - \frac{z^3}{3!} + \ldots\right)\left(1 - z + z^2 - \ldots\right)^2 \\ &= z^{-2}\left(1 - \frac{z^2}{6} + \ldots\right)\left(1 - 2z + 3z^2 + \ldots\right) \\ &= z^{-2}\left(1 - 2z + \left(3 - \frac{1}{6}\right)z^2 + \ldots\right) \\ &= z^{-2} - 2z^{-1} + \left(3 - \frac{1}{6}\right) + \ldots\end{aligned}$$

The principal part is $z^{-2} - 2z^{-1}$