Question

Let C_s denote the hyperbolic circle in the Poincaré disc $\mathbf{D}$ with hyperbolic radius s and hyperbolic center 0. Calculate the circumference of C_s as a function of s.

Answer

If the hyperbolic radius is s, then the Euclidean radius is $r = \tanh(\frac{s}{2})$. Parametrize the circle by $f(t) = re^{-it}$ with $0 \leq t \leq 2\pi$.

$$\begin{aligned}
\text{length}_{\mathbf{D}}(f) &= \int_0^{2\pi} \frac{2r\,dt}{1-r^2} \\
&= \frac{4\pi r}{1-r^2} \\
&= \frac{4\pi \tanh(\frac{s}{2})}{1-\tanh^2(\frac{s}{2})} \cdot \frac{\cosh^2(\frac{s}{2})}{\cosh^2(\frac{s}{2})} \\
&= \frac{4\pi \sinh(\frac{s}{2})\cosh(\frac{s}{2})}{\cosh^2(\frac{s}{2}) - \sinh^2(\frac{s}{2})} \\
&= 4\pi \sinh(\frac{s}{2})\cosh(\frac{s}{2}) \\
&= 2\pi \sinh(s)
\end{aligned}$$

(Here we use two identities:

- $\cosh^2(x) - \sinh^2(x) = 1$
- $2\sinh(\frac{x}{2})\cosh(\frac{x}{2}) = \sinh(x)$)