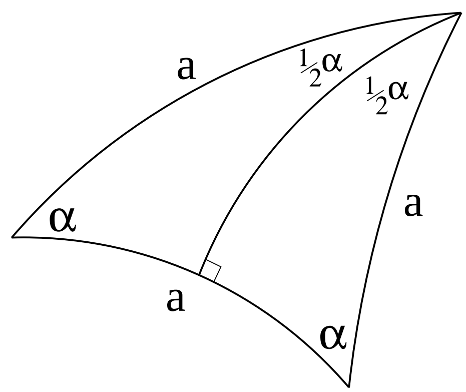
Question

Show that a triangle T in $\mathbf{H}$ is an equilateral triangle (that is, all sides have the same length) if and only if its angles are all equal.

Now, let α be the angle at a vertex of T and let a be the hyperbolic length of a side of T. Show that $2\cosh(\frac{1}{2}a)\sin(\frac{1}{2}\alpha) = 1$.

Answer



by lcI: all angles are equal and have $\cos(\alpha) = \dfrac{\cosh^2(a) - \cosh(a)}{\sinh^2(a)}$ (and the fact that an angle in the range $[0, \pi]$ is completely determined by its cosine) (the converse, that equal angles imply equal side lengths follows immediately from lcII, with

$$\cosh(a) = \frac{\cos(\alpha) - \cos^2(\alpha)}{\sin^2(\alpha)})$$

The bisector of the angles intersects the opposite side in a right angle by a geometric argument, namely the triangle is taken to itself by reflection in the bisecting line. The same argument shows that the bisecting line intersects the opposite side in its midpoint.

Now use lcII. $\cos(\frac{1}{2}\alpha) = -\cos(\alpha)\cos(\frac{\pi}{2}) + \sin(\alpha)\sin(\frac{\pi}{2})\cosh(\frac{1}{2}a)$

$\cosh(\frac{1}{2}a) = \dfrac{\cos(\frac{1}{2}\alpha)}{\sin(\alpha)} = \dfrac{1}{2\sin(\frac{1}{2}\alpha)}$

(since $\sin(\alpha) = 2\sin(\frac{1}{2}\alpha)\cos(\frac{1}{2}\alpha)$)

and so $2\cosh(\frac{1}{2}a)\sin(\frac{1}{2}\alpha) = 1$ as desired.