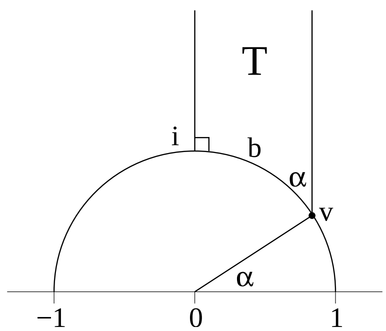
Question

Let T be a triangle in the hyperbolic plane with angles α, 0, and $\frac{\pi}{2}$. Prove that $\cosh(b)\sin(\alpha) = 1$, where b is the hyperbolic length of the side of T opposite the vertex with angle 0. [Note that since the triangle is not compact, the hyperbolic trigonometric rules as we have stated and proved them in class do not apply.]

Answer

Using the transitivity properties of $\text{Möb}^+(\mathbf{H})$, assume that the vertex with angle 0 is at ∞, the vertex with angle $\dfrac{\pi}{2}$ is at i, and the side joining the vertices with angles $\dfrac{\pi}{2}$ and α lies on the unit circle S_1, as pictured below:



$$v = e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

$$\begin{aligned}
1 + \frac{|v-i|^2}{2\cdot 1\cdot \sin(\alpha)} &= 1 + \frac{\cos^2(\alpha) + (\sin(\alpha) - 1)}{2\sin(\alpha)} \\
&= \frac{2\sin(\alpha) + 2 + -2\sin(\alpha)}{2\sin(\alpha)} \\
&= \frac{1}{\sin(\alpha)} = \csc(\alpha)
\end{aligned}$$

So, by previous problem, $\cosh(b) = \csc(\alpha)$ and so $\cosh(b)\sin(\alpha) = 1$ as desired.