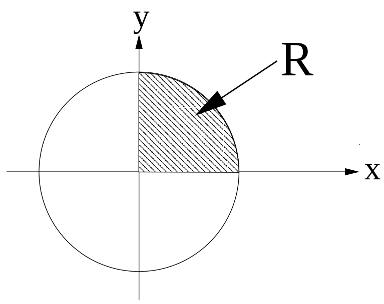
Question

If R is the region defined by the inequalities $x^2+y^2 \leq 1$, $x \geq 0$ and $y \geq 0$, evaluate the double integral

$$\iint_R (xy+1)\,d(x,y)$$

by first transforming it into plane polar co-ordinates.

Answer



R is defined by the r, θinequalities

$$\begin{array}{ccccc} 0 & \leq & r & \leq & 1 \\ 0 & \leq & \theta & \leq & \frac{\pi}{2} \end{array}$$

Since $x = r\cos\theta$ and $y = r\sin\theta$ we have:

$$xy = (r\cos\theta)(r\sin\theta) = r^2\sin\theta\cos\theta = \frac{1}{2}r^2\sin 2\theta$$

The integral becomes:

$$\begin{aligned}
\iint_R (xy+1)\,d(x,y) &= \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1}\left(\frac{1}{2}r^2\sin 2\theta + 1\right) r\,drd\theta \\
&= \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1}\frac{1}{2}r^3\sin 2\theta\,drd\theta + \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} r\,drd\theta \\
&= \left\{\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta\,d\theta\right\}\left\{\int_0^1 r^3\,dr\right\} + \left\{\int_0^{\frac{\pi}{2}} d\theta\right\}\left\{\int_0^1 r\,dr\right\} \\
&\ \text{(since all limits are independent of } r \text{ and } \theta\text{)} \\
&= \left[-\frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^1 + [\theta]_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^1 \\
&= \left(\frac{1}{4}+\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}+\frac{\pi}{8}
\end{aligned}$$