Question

For each of the following functions described below, use the Intermediate value property for continuous functions to determine whether there is a solution to the given equation in the specified set.

1. $f(x) = x$, where $f(x)$ is continuous on the closed interval $[a, b]$ and satisfies $f(a) < a < b < f(b)$ for all $x \in [a, b]$;
2. $g(x) = 0$, where $g(x) = x^2 - \cos(x)$;
3. $f(x) = 0$ on the interval $[-a, a]$, where a is an arbitrary positive real number and $f(x) = x^{1995} + 7654x^{123} + x$;
4. $\tan(x) = e^{-x}$ for x in $[-1, 1]$;
5. $x^3 + 2x^5 + (1 + x^2)^{-2} = 0$ for x in $[-1, 1]$;
6. $3\sin^2(x) = 2\cos^3(x)$ for $x > 0$;
7. $3 + x^5 - 1001x^2 = 0$ for $x > 0$;

Answer

1. as before, consider the continuous function $g(x) = f(x) - x$. Since $f(a) < a$, we have that $g(a) = f(a) - a < 0$. Since $f(b) > b$, we have that $g(b) = f(b) - b > 0$. Hence, the intermediate value property applied to g yields that there exists c in (a, b) with $g(c) = 0$. That is, $f(c) - c = 0$, and so $f(c) = c$. Hence, the equation $f(x) = x$ has a solution in $[a, b]$.
2. first of all, note that $g(x) = x^2 - \cos(x)$ is continuous on all of $\mathbf{R}$, and so is continuous on every closed interval $[a, b]$ in $\mathbf{R}$. In order to apply the intermediate value property to find a point c at which $g(c) = 0$, we need to find a and b so that $g(a) > 0$ and $g(b) < 0$ (or vice versa), and the intermediate value property then implies the existence of such a number c between a and b.

   So, let's start plugging numbers into g: $g(0) = -\cos(0) = -1 < 0$ and $g(2) = (2)^2 - \cos(2) = 4.6536... > 0$, and so there exists a number c_1 between 0 and 2 with $g(c_1) = 0$. (Note that since $(2)^2 = (-2)^2$ and $\cos(2) = \cos(-2)$, we also have that there exists c_2 between -2 and 0 with $g(c_2) = 0$.)

3. for $f(x) = x^{1995} + 7654x^{123} + x$ on the closed interval $[-a, a]$, start by verifying continuity; actually, f is continuous on all of $\mathbf{R}$ being a polynomial, and hence is continuous on $[-a, a]$. Now, check the sign of f on the endpoints of the given interval: $f(a) = a^{1995} + 7654a^{123} + a > 0$ (since $a > 0$) and $f(-a) = (-a)^{1995} + 7654(-a)^{123} + (-a) = -f(a) < 0$, and so the intermediate value property implies that there exists some c in $(-a, a)$ with $f(c) = 0$. (And actually, casual inspection reveals that $f(0) = 0$.)

4. for $\tan(x) = e^{-x}$ for x in $[-1, 1]$, start by defining $g(x) = \tan(x) - e^{-x}$, so that $\tan(c) = e^{-c}$ if and only if $g(c) = 0$, as was done above. Note that g is continuous on $[-1, 1]$, since e^{-x} is continuous on all of $\mathbf{R}$ and $\tan(x)$ is continuous as long as its denominator $\cos(x)$ is non-zero, which holds true on $[-1, 1]$. Since we are working on the closed interval $[-1, 1]$, check the values of g on the endpoints: $g(1) = \tan(1) - e^{-1} = 1.1895... > 0$ and $g(-1) = -4.2757... < 0$, and so there exists some c in $(-1, 1)$ with $g(c) = 0$, and hence with $\tan(c) = e^{-c}$.

5. as above, $f(x) = x^3 + 2x^5 + (1 + x^2)^{-2}$ is continuous on $[-1, 1]$, as it is the sum of a polynomial and a rational function whose denominator is non-zero on $[-1, 1]$. As always, check the endpoints of the interval first: $f(1) = \frac{13}{4}$ and $f(-1) = -\frac{11}{4}$, and so by the intermediate value property, there is some c in $(-1, 1)$ at which $f(c) = 0$.

6. consider $f(x) = 3\sin^2(x) - 2\cos^3(x)$. Since both $\sin(x)$ and $\cos(x)$ are continuous on all of $\mathbf{R}$, we have that f is continuous on all of $\mathbf{R}$. Since no specific closed interval is given, we need to find an appropriate interval on which to apply the intermediate value property for f, if in fact such an interval exists. Fortunately, we remember that $\sin(k\pi) = 0$ for all integers k, and so we may consider the interval $[k\pi, (k+1)\pi]$ for any integer $k \geq 1$, so that the interval lies in $(0, \infty)$. At the endpoints of this interval, $f(k\pi) = -2\cos^3(k\pi)$ and $f((k+1)\pi) = -2\cos^3((k+1)\pi)$. Since $\cos(k\pi)$ and $\cos((k+1)\pi)$ are equal to ± 1 and have opposite signs, $f(k\pi)$ and $f((k+1)\pi)$ are both non-zero and have opposite signs, and so by the intermediate value property, there is a point c_k in $(k\pi, (k+1)\pi)$ at which $f(c_k) = 0$, that is, at which $3\sin^2(c_k) = 2\cos^3(c_k)$, as desired.

7. first, note that $f(x) = 3 + x^5 - 1001x^2$ is a polynomial and so is continuous on all of $\mathbf{R}$, and in particular is continuous for $x > 0$. As above, we need to choose a closed interval on which to apply the intermediate value property. Let's start by evaluating f at some of the natural numbers: $f(1) = -997$; $f(2) = -3969$; $f(10) = -90097$;

$f(11) = 880$. Hence, the intermediate value property implies that there is a number c in the open interval $(10, 11)$ at which $f(c) = 0$.